THE NORM PROPERTIES OF THE RESTRICTIONS OF A COMPACT OPERATOR

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PREFACE

In 1954 Aronszajn and Smith showed that a compact operator on an infinite - dimensional complex Banach space has a non - trivial closed invariant subspace. Their method of proof was simplified by Bonsall who also demonstrated that the theorem is true in normed linear spaces. The basis of this thesis is the following strengthened version of this result :

<u>Theorem</u>: Let S be a compact operator on an infinite - dimensional normed linear space and let $\kappa > 0$. Then there exists a non - zero closed subspace F invariant for S with the norm of the restriction of S to F less than or equal to κ .

We shall call this theorem the Main Theorem .

The layout of this thesis is as follows. In Chapter I we introduce the necessary basic definitions and results. Some of the standard theory of linear maps on finite - dimensional spaces is stated together with the relevant parts of the Riesz - Schauder theory of compact operators. This theory is discussed for both real and complex cases. There are many suitable references for the complex case and for this case results are stated without proof. However, as we were unable to find references for the results on real spaces, proofs are given in this case .

Chapter II contains the proof of the Main Theorem and some corollaries of it are also given. One of these corollaries is the Aronszajn - Smith Theorem stated above .

In Chapter III we give examples of the Main Theorem and show that it does not hold in general for polynomially compact operators. This is of special interest since it has been shown by Bernstein and Robinson and others that polynomially compact operators on infinite - dimensional normed linear spaces have non - trivial invariant subspaces .

Given a class of compact operators we may ask under what circumstances a generalised version of the Main Theorem would hold for this class. In Chapter IV we obtain some results on this topic. Also, a version of the Main Theorem is proved for locally convex topological vector spaces .

It has been shown by Ringrose that the spectral properties of a compact operator can be deduced from the behaviour of the operator on a maximal chain of invariant subspaces. In Chapter V we scrutinise the behaviour of the norm of a compact operator when restricted to a set of subspaces forming a chain. In particular, we define an idea of continuity of the norm in this context and obtain results on when this property holds. The best results hold when we deal with reflexive spaces .

I would like to take this opportunity of recording my sincere thanks to my research supervisors Dr. J.C. Alexander and Dr. T.A. Gillespie for all their help and guidance .

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CHAPTER 1

PRELIMINARIES

The purpose of this chapter is to define some of the terms required for this thesis and to state well-known results about linear maps on finite-dimensional spaces and compact operators on normed linear spaces. The only results which are not standard concern real spaces and for these we give outline proofs .

1. Normed linear spaces.

Let X be a normed linear space. The field of scalars will be taken to be either the real numbers $\underline{R}$ or the complex numbers $\underline{C}$ unless restricted to one or the other .

A subspace of X is a non-empty linear subset which is closed in the norm topology. We denote by (0) the subspace consisting of the zero vector alone .

If D is a subset of X we denote by $\overline{D}$ or $\mathrm{cl}D$ the closure of D in the norm topology. $\mathrm{sp}D$ is the set of finite linear combinations of elements of D and we use $\overline{\mathrm{sp}}D$ as short-hand for $\mathrm{cl}(\mathrm{sp}D)$. Clearly if D is non-empty then $\overline{\mathrm{sp}}D$ is a subspace of X .

A non-trivial subspace of X is a subspace which is not equal to (0) or X . We shall use the symbol $\subset$ to denote strict inclusion. Thus a non-trivial subspace M of X satisfies $(0) \subset M \subset X$.

Let M be a subspace of X and let $x \in X$. We define

$$d(x,M) = \inf \{ \| x - y \| : y \in M \} ;$$

we call this the <u>distance</u> from x to M. If M is finite-dimensional it follows from the compactness of closed bounded sets in finite-dimensional spaces that there is a $y \in M$ such that

$$d(x,M) = \| x - y \| .$$

Such a y is called a <u>nearest</u> <u>vector</u> in M to x.

An <u>$\underline{N}$-subsequence</u> is an infinite subset of the set of natural numbers $\underline{N}$. We shall use the Greek letters μ and ν for $\underline{N}$-subsequences.

Let μ be an $\underline{N}$-subsequence and let $\{ x_n \}_{n \in \mu}$ be a subset of X indexed by the elements of μ. If there is an $x \in X$ with the property that, for any $\eta > 0$, there exists an $n_0 \in \mu$ such that

$$\| x_n - x \| < \eta$$

for $n \in \mu$ and $n \geq n_0$, then we say that the limit of $\{ x_n \}_{n \in \mu}$ exists and is equal to x. We write this as $\lim\limits_{n \in \mu} x_n = x$, or, if no confusion can arise, $\lim\limits_{\mu} x_n$.

Let μ be an $\underline{N}$-subsequence and let $\{ H_n \}_{n \in \mu}$ be a set of subspaces of X indexed by μ. We define the set $\liminf\limits_{\mu} H_n$ by the equation,

$$\liminf_{\mu} H_n = \{ x \in X : \lim_{\mu} d(x,H_n) = 0 \} .$$

Clearly an alternative definition is:

$x \in \liminf\limits_{\mu} H_n$ if and only if there exists a subset $\{ x_n \}_{n \in \mu}$ of X indexed by μ with $x_n \in H_n$ $(n \in \mu)$ and $\lim\limits_{\mu} x_n = x$.

It is a simple consequence of the definition that $\liminf_{\mu} H_n$ is itself a subspace .

The set of bounded linear maps from X to itself is denoted by $B(X)$. The elements of $B(X)$ will be called <u>operators</u> on X.

Let $T \in B(X)$. A subspace M for which $TM \subseteq M$ is called an <u>invariant</u> subspace for T .

Let Λ be a subset of $B(X)$. Lat Λ is defined to be the set of those subspaces of X which are invariant for all operators in Λ . Lat Λ is closed under the operations of closed span and intersection; also (0) , $X \in$ Lat Λ . For a single operator T we denote Lat $\{T\}$ as Lat T .

We denote by ker T the set $\{ x \in X : Tx = 0 \}$.

Let M be a subspace of X and $x \in X$. We denote by $x + M$ the coset of M which contains x . The set of all such cosets is denoted by X/M . When endowed with the canonical linear operations and a norm given by $\| x + M \| = d(x,M)$, X/M becomes a normed linear space. The <u>codimension</u> of M in X is defined to be the dimension of X/M .

Let $T \in B(X)$ and $M \in$ Lat T . We define linear maps $T|M$ and T_M on M and X/M respectively by the relations

$$(T|M)x = x, \ (x \in M)$$

$$T_M(x + M) = Tx + M \quad (x + M \in X/M) \ ,$$

$T|M$ is known as the <u>restriction</u> of T to M . Clearly $T|M \in B(M)$ and $T_M \in B(X/M)$. Even if $M \notin$ Lat T we shall use the expression $\| T|M \|$ to denote $\sup \{ \| Tx \| : x \in M , \| x \| \leq 1 \}$.

For the remainder of this section, let X be a real normed linear space .

We denote by $\widetilde{X}$ the <u>complexification</u> of X . This, by definition, is the set of formal sums $x + iy$ where x , y run through the set X .

Let $x , y , x_1 , y_1 , x_2 , y_2 \in X$ and $\alpha , \beta \in \underline{R}$. We define equality, addition, scalar multiplication, and norm on $\widetilde{X}$ by:

(i) $x_1 + iy_1 = x_2 + iy_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$;

(ii) $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$;

(iii) $(\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\beta x + \alpha y)$;

(iv)

$$\| x + iy \| = \sup_{\theta \in \underline{R}} [\| \cos\theta\, x - \sin\theta\, y \| + \| \sin\theta\, x + \cos\theta\, y \|] .$$

With these definitions $\widetilde{X}$ becomes a complex normed linear space and we have

$$\| x \| + \| y \| \leqslant \| x + iy \| \leqslant \sqrt{2} (\| x \| + \| y \|) \quad (x , y \in X).$$

Let M be a subspace of $\widetilde{X}$. We define the <u>conjugate space</u> M^* of M as follows:

$$M^* = \{ x - iy : x , y \in X , x + iy \in M \} .$$

This is also a subspace .

We say that M is <u>self-conjugate</u> if $M = M^*$. Clearly, if N is any subspace then $N \cap N^*$ is self-conjugate and if, in addition, N is finite-dimensional then $N + N^*$ is self-conjugate .

Let M be a self-conjugate subspace of $\widetilde{X}$. We define

$$\operatorname{Re} M = \{ x \in X : x + i0 \in M \} .$$

This is also a subspace of X. The importance of this definition is that M is the complexification of $\mathrm{Re}\, M$ and so $\dim_{\underline{R}} \mathrm{Re}\, M = \dim_{\underline{C}} M$.

Let $T \in B(X)$. We define on $\tilde{X}$ the <u>complexification</u> $\tilde{T}$ of T by

$$\tilde{T}(x+iy) = Tx + iTy \quad (x , y \in X) .$$

Clearly $\tilde{T} \in B(\tilde{X})$ and

$$\| T \| \leq \sqrt{2} \, \| \tilde{T} \| \leq 2 \, \| T \| .$$

We note the following : for $\lambda \in \underline{C}$ and m a positive integer, we have that $(\tilde{T} - \lambda^*)^m \tilde{X}$ is the conjugate of $(\tilde{T} - \lambda)^m \tilde{X}$ and $\ker (\tilde{T} - \lambda^*)^m$ is the conjugate of $\ker (\tilde{T} - \lambda)^m$.

2. Finite-dimensional spaces.

In this section the terms 'operator' and 'subspace' are taken to have purely algebraic connotations .

The following theorem may be found in [12 p.106] :

(1,2,1) <u>Theorem</u>: Let E be a complex vector space of dimension $n < \infty$ and let T be an operator on E. Then there exist subspaces $M_0 , M_1 , \ldots , M_n$ of E with :

(i) $(0) = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = E$;

(ii) $\dim M_i = i$, $i = 0 , 1 , 2 , \ldots , n$;

(iii) $TM_i \subseteq M_i$, $i = 0 , 1 , 2 , \ldots , n$.

We need a corresponding result for real vector spaces. The following result seems to be well-known and it appears in [9] . We shall, however, sketch a proof of it .

(1.2.2) Theorem: Let E be a real vector space of dimension $n < \infty$ and let S be an operator on E. Then there exist an integer m and subspaces $N_0, N_1, \ldots, N_m$ of E with :

(i) $\frac{1}{2}n \leq m \leq n$;

(ii) $(0) = N_0 \subset N_1 \subset N_2 \subset \ldots \subset N_m = \mathbf{E}$;

(iii) $1 \leq \dim N_{i+1} - \dim N_i \leq 2$ for $i = 0, 1, \ldots, m-1$;

(iv) $SN_i \subseteq N_i$, $i = 0, 1, \ldots, m$.

Proof: Let $\widetilde{E}$, $\widetilde{S}$ denote the complexifications of E, S respectively. Since $\dim_{\underline{C}} \widetilde{E} = n$, it follows from Theorem (1.2.1) that there are subspaces $M_0, M_1, \ldots, M_n$ of $\widetilde{\mathbf{E}}$ with the following properties :

(i) $(0) = M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_n = \widetilde{\mathbf{E}}$;

(ii) $\dim M_i = i$, $i = 0, 1, \ldots, n$;

(iii) $\widetilde{S}M_i \subseteq M_i$, $i = 0, 1, \ldots, n$.

Consider the self-conjugate subspaces :

$$M_0 + M_0^*, M_1 + M_1^*, M_2 + M_2^*, \ldots, M_n + M_n^* .$$

Obviously they form a chain and $M_0 + M_0^* = (0)$, $M_n + M_n^* = \widetilde{E}$. A dimension argument gives that

$$0 \leq \dim (M_{i+1} + M_{i+1}^*) - \dim (M_i + M_i^*) \leq 2 ,$$

for $i = 0, 1, \ldots, n-1$. Also $\widetilde{S} (M_i + M_i^*) \subseteq M_i + M_i^*$ for $i = 0, 1, \ldots, n$. Thus if we let m be the number of distinct members in the chain and let $N_0, N_1, \ldots, N_m$ be the real parts of the distinct members, retaining the same order, then these subspaces have the required properties .

3. Compact operators.

We need some of the Riesz - Schauder theory of compact operators. For complex normed linear spaces the theory is well known and may be found in the standard references eg. [7 , 21]. Though the theory appears to be fairly well-known for real spaces, we were unable to find references for it. We shall, therefore, sketch in proofs using similar techniques to the last section .

(1.3.1) Definition: Let S be a compact operator on a complex normed linear space X . We define $\sigma_0(S)$ to be the set of non - zero eigen - values of S .

We shall use $\underline{C}_0$ to denote the set $\{ \lambda \in \underline{C} : \lambda \neq 0 \}$. With this notation an equivalent definition of $\sigma_0(S)$ is :

$$\sigma_0(S) = \{ \lambda \in \underline{C}_0 : \ker (S - \lambda) \neq (0) \} .$$

(1.3.2) Theorem: Let S be a compact operator on a complex normed linear space X . Then $\sigma_0(S)$ is countable. If $\sigma_0(S)$ is infinite, then 0 is the only accumulation point of $\sigma_0(S)$.

(1.3.3) Lemma: Let S be a compact operator on a complex normed linear space X ,and let $\lambda \in \underline{C}_0$. Then there is a non - negative integer m such that

$$\ker [(S - \lambda)^{m+1}] = \ker [(S - \lambda)^m] .$$

Moreover for such an m ,

$$\ker [(S - \lambda)^{m+k}] = \ker [(S - \lambda)^m]$$

for $k = 0 , 1 , 2 , \ldots$.

(1.3.4) Definition: Let S be a compact operator on a complex normed linear space X and let $\lambda \in \underline{C}_0$. We define the index of λ for S to be the smallest non-negative integer m for which

$$\ker [(S - \lambda)^{m+1}] = \ker [(S - \lambda)^m] .$$

(1.3.5) Theorem: Let S be a compact operator on a complex normed linear space X. Let $\lambda \in \underline{C}_0$ and let m be the index of λ for S. If $M = (S - \lambda)^m X$ and $N = \ker (S - \lambda)^m$ then :

(i) M is a subspace of X ;

(ii) N is a finite-dimensional subspace and M is of finite codimension in X ;

(iii) $M \oplus N = X$;

(iv) $SM \subseteq M$, $SN \subseteq N$;

(v) $(S - \lambda)|M$ is invertible. In particular λ is not an eigen-value of $S|M$;

(vi) $(S - \lambda)|N$ is nilpotent .

We turn now to the study of compact operators on real normed linear spaces. The following lemma allows us to apply the preceding results to the real case.

(1.3.6) Lemma: The complexification of a compact operator on a real normed linear space is a compact operator .

(1.3.7) Theorem: Let S be a compact operator on a real normed linear space X and let $\widetilde{X}$, $\widetilde{S}$ denote the complexifications of X , S respectively. Let $\lambda \in \underline{C}_0$. Then there exist subspaces M and N of X with the following properties :

(i) M is of finite codimension in X , N is finite-dimensional ;

(ii) $M \oplus N = X$;

(iii) $SM \subseteq M$, $SN \subseteq N$;

(iv) λ , $\lambda^* \notin \sigma_o(\widetilde{S}|\widetilde{M})$ where $\widetilde{M}$ is the complexification of M .

__Proof:__ Suppose that m is the index of λ for $\widetilde{S}$. Let

$$M = \mathrm{Re}\ [\ (\widetilde{S} - \lambda)^m \widetilde{X} \cap (\widetilde{S} - \lambda^*)^m \widetilde{X}\ ]$$

and let

$$N = \mathrm{Re}\ [\ \ker\ (\widetilde{S} - \lambda)^m + \ker\ (\widetilde{S} - \lambda^*)^m\ ]\ .$$

It is not difficult to check that M and N have the desired properties .

CHAPTER II

THE MAIN THEOREM

One of the fundamental problems of the theory of linear operators is to find an answer to the following question :

Let X be an infinite - dimensional complex Banach space and let T be an arbitrary operator on X . Does T have a non-trivial invariant subspace ?

The answer is, in general, not known. However, for certain classes of operators an affirmative answer has been found. Our interest is with compact operators which were shown by Aronszajn and Smith [1] in 1954 to have non - trivial invariant subspaces. They attribute the result for Hilbert space to von Neumann. Since then, various invariant subspace theorems about operators related to compact operators have been proved. A selection of these appears in Section 2 of Chapter IV .

In this chapter we show that an invariant subspace of a compact operator on an infinite - dimensional space can be chosen satisfying a condition on the norm of the restriction. We state this theorem here and we shall call it the Main Theorem .

Theorem: Let S be a compact operator on an infinite - dimensional real or complex normed linear space X , and let $\kappa > 0$. Then S has a non - zero invariant subspace F with $\| S|F \| \leqslant \kappa$.

To prove this theorem we first prove a special case of it and then use Theorems (1.3.5) and (1.3.7) to give full generality.

1. The case where S has no eigen-values.

(2.1.1) **Theorem:** Let S be a compact operator on an infinite-dimensional real or complex normed linear space X, and let $\kappa > 0$. Suppose that S has no non-zero finite-dimensional invariant subspaces. Then S has a non-zero invariant subspace F with $\| S|F \| \leqslant \kappa$.

Proof: Let e be a non-zero vector in X and let

$$N = \overline{\mathrm{sp}}\, \{ e , Se , S^2 e , \ldots \ldots \} .$$

N is a non-zero invariant subspace of S so if $\| S|N \| \leqslant \kappa$ we have nothing further to prove. So assume

$$\| S|N \| > \kappa . \tag{1}$$

For each $n \geqslant 1$, let $E_n = \mathrm{sp}\, \{ e , Se , \ldots \ldots , S^{n-1}e \}$. Since S has no finite-dimensional invariant subspaces the set $\{ e , Se , \ldots \ldots , S^{n-1}e \}$ is linearly independent, so $\dim E_n = n$.

Since E_n is finite-dimensional, we can find an f_n in E_n which is nearest to $S^n e$. That is,

$$d(S^n e, E_n) = \| S^n e - f_n \| . \tag{2}$$

For each $n \geqslant 1$, define S_n to be the linear map on E_n with

$$S_n(S^r e) = S^{r+1} e \ , \ r = 0 , 1 , \ldots , n-2 ,$$

and

$$S_n(S^{n-1} e) = f_n .$$

Let $x \in E_n$; then $x = \lambda S^{n-1}e + y$ where $\lambda \in \underline{C}$ and $y \in E_{n-1}$. We have

$$Sx = \lambda S^n e + Sy$$

and

$$S_n x = \lambda f_n + Sy \ .$$

Thus

$$\| Sx - S_n x \| = |\lambda| \ \| S^n e - f_n \| = |\lambda| \ d(S^n e, E_n) \ .$$

But

$$|\lambda| \ d(S^n e, E_n) = d(\lambda S^n e, E_n) = d(\lambda S^n e + Sy, E_n) = d(Sx, E_n) \ .$$

So we have, for all x in E_n ,

$$\| Sx - S_n x \| = d(Sx, E_n) \ . \tag{3}$$

Suppose that for each $n \geq 1$ we have a subspace H_n of E_n which is invariant for S_n . If μ is an $\underline{N}$-subsequence then

$$\liminf_{\mu} SH_n \subseteq \liminf_{\mu} H_n \ . \tag{4}$$

For, let $y \in \liminf_{\mu} SH_n$; then $y = \lim_{\mu} Sx_n$ where $x_n \in H_n$ for each $n \in \mu$. Now

$$\begin{aligned} \| y - S_n x_n \| &\leq \| y - Sx_n \| + \| Sx_n - S_n x_n \| \\ &= \| y - Sx_n \| + d(Sx_n, E_n) \\ &\leq \| y - Sx_n \| + \| y - Sx_n \| + d(y, E_n) \ . \end{aligned}$$

Since $y \in N$ and $N = \liminf_{\mu} E_n$, it follows that $\lim_{\mu} d(y, E_n) = 0$ and so $\lim_{\mu} \| y - S_n x_n \| = 0$. But $S_n x_n \in H_n$ for all $n \in \mu$, so $y \in \liminf_{\mu} H_n$.

It follows from the continuity of S that

$$S(\liminf_{\mu} H_n) \subseteq \liminf_{\mu} SH_n \tag{5}$$

for any $\underline{N}$ - subsequence μ , and thus that $\liminf_{\mu} H_n$ is an invariant subspace for S .

From (1) and the fact that $N = \mathrm{cl} \bigcup_{n=1}^{\infty} E_n$, we deduce that there exists an $n_o \in \underline{N}$ such that, for $n \geq n_o$,

$$\| S|E_n \| > \kappa . \tag{6}$$

Since E_n is finite - dimensional and S_n is a linear map on E_n , there exist by Theorems (1.2.1) and (1.2.2) an integer m_n and subspaces $E_n^0 , E_n^1 , \ldots , E_n^{m_n}$ of E_n with the following properties :--

If X is a complex normed linear space :

(i) $m_n = n$;

(ii) $E_n^i \subset E_n^{i+1}$, $i = 0 , 1 , \ldots , n-1$;

(iii) $\dim E_n = i$, $i = 0 , 1 , \ldots , n$;

(iv) $S_n E_n^i \subseteq E_n^i$, $i = 0 , 1 , \ldots , n$.

If X is a real normed linear space :--

(i) $\frac{1}{2}n \leq m_n \leq n$;

(ii) $E_n^0 = (0)$, $E_n^{m_n} = E_n$;

(iii) $E_n^i \subset E_n^{i+1}$, $i = 0 , 1 , \ldots , m_n - 1$;

(iv) $1 \leq \dim (E_n^{i+1}) - \dim (E_n^i) \leq 2$,

$i = 0 , 1 , \ldots , m_n - 1$;

(v) $S_n E_n^i \subseteq E_n^i$, $i = 0 , 1 , \ldots , m_n$.

In both the real and complex cases the sequence of numbers

$$0 = \| S|E_n^0 \| , \| S|E_n^1 \| , \| S|E_n^2 \| , \ldots , \| S|E_n^{m_n} \|$$

is an increasing sequence and for each $n \geq n_o$ we have from (6) that $\| S|E_n^{m_n} \| > \kappa$. So, provided $n \geq n_o$, we can choose F_n , G_n from the chain $\{ E_n^i \}_{i=0}^{m_n}$ with the following properties :

$F_n \subset G_n$; (7)

for complex X : $\dim G_n = 1 + \dim F_n$; (8)

for real X : $1 \leq \dim G_n - \dim F_n \leq 2$; (9)

$S_n F_n \subseteq F_n$, $S_n G_n \subseteq G_n$; (10)

$\| S | F_n \| \leq \kappa$; (11)

$\| S | G_n \| > \kappa$. (12)

By (12) , for each $n \geq n_o$, there exists an $x_n \in G_n$ with $\| x_n \| = 1$ and $\| Sx_n \| > \kappa$. Using the compactness of S , there exists a subsequence μ such that $y = \lim_\mu Sx_n$ exists. Obviously $\| y \| \geq \kappa$ so $y \neq 0$ and, by definition, $y \in \liminf_\mu SG_n$. Now, by (10) and (4), $\liminf_\mu SG_n$ is contained in $\liminf_\mu G_n$. Thus $\liminf_\mu G_n \neq (0)$.

Let $G = \liminf_\mu G_n$. By (4) and (5), G is a non - zero invariant subspace for S . It follows from the hypotheses that G is infinite - dimensional .

We now consider the complex and real cases separately to prove that there exists a subsequence ν such that

$$\liminf_\nu F_n \neq (0) .$$

When X is a complex normed linear space :--

Let u , v be two linearly independent vectors in G . Then $u = \lim_\mu u_n$, $v = \lim_\mu v_n$ where u_n , $v_n \in G_n$ for each $n \in \mu$. By (7) and (8) , there exist, for each $n \in \mu$ complex numbers α_n , β_n, not both zero, such that

$$\alpha_n u_n + \beta_n v_n \in F_n .$$

We may assume that $|\alpha_n| + |\beta_n| = 1$ for each $n \in \mu$. Using the relative compactness of bounded sets in the complex plane, we may find an $\underline{N}$ - subsequence ν with $\nu \subseteq \mu$ such that $\alpha = \lim_{\nu} \alpha_n$, $\beta = \lim_{\nu} \beta_n$ exist. Then

$$\alpha u + \beta v = \lim_{\nu} (\alpha_n u_n + \beta_n v_n) \in \liminf_{\nu} F_n .$$

Now $|\alpha| + |\beta| = 1$ and so, since u , v are linearly independent, $\alpha u + \beta v \neq 0$. Therefore

$$\liminf_{\nu} F_n \neq (0) .$$

When X is a real normed linear space :--

Let u , v , w be three linearly independent vectors in G . Then $u = \lim_{\mu} u_n$, $v = \lim_{\mu} v_n$, $w = \lim_{\mu} w_n$ where u_n , v_n , w_n are elements of G_n ,for each $n \in \mu$. By (7) and (9) there exist, for each $n \in \mu$, real numbers α_n , β_n , γ_n, not all zero, such that $\alpha_n u_n + \beta_n v_n + \gamma_n w_n \in F_n$. We may assume that $|\alpha_n| + |\beta_n| + |\gamma_n| = 1$. Let ν be an $\underline{N}$ - subsequence with $\nu \subseteq \mu$ such that $\alpha = \lim_{\nu} \alpha_n$, $\beta = \lim_{\nu} \beta_n$, $\gamma = \lim_{\nu} \gamma_n$ exist. Then

$$\alpha u + \beta v + \gamma w = \lim_{\nu} (\alpha_n u_n + \beta_n v_n + \gamma_n w_n) \in \liminf_{\nu} F_n .$$

Now $|\alpha| + |\beta| + |\gamma| = 1$ and so, since u , v , w are linearly independent $\alpha u + \beta v + \gamma w \neq 0$. Therefore

$$\liminf_{\nu} F_n \neq (0) .$$

For both cases, let $F = \liminf_{\nu} F_n$. By (4) and (5) F is invariant for S and by the above results it is non - zero . Let $x \in F$, ie. $x \in \liminf_{\nu} F_n$. Then $x = \lim_{\nu} x_n$ where $x_n \in F_n$ for all $n \in \nu$. By (11) , $\| Sx_n \| \leq \kappa \| x_n \|$ for all $n \in \nu$ and so

$$\| Sx \| = \lim_{\nu} \| Sx_n \| \leq \kappa \lim_{\nu} \| x_n \| = \kappa \| x \| .$$

Thus $\| S|F \| \leq \kappa$ and so F has all the desired properties .

(2.1.2) Corollary: (Aronszajn and Smith [1]) Let X be a complex normed linear space of dimension greater than or equal to two, and let S be a compact operator on X . Then S has a non - trivial invariant subspace .

Proof: If X is finite - dimensional the result is contained in Theorem (1.2.1) . So assume X is infinite - dimensional. If S has an eigen - value then sp $\{ x_0 \}$ is a non - trivial invariant subspace where x_0 is a (non - zero) eigen - vector. If S has no eigen - values then $\| S \| > 0$. Let κ be a real number with $0 < \kappa < \| S \|$. We now apply Theorem (2.1.1) to obtain a non - zero subspace F which is invariant for S with $\| S|F \| \leq \kappa$. Obviously $F \neq X$ and so we have the required subspace .

(2.1.3) Corollary: (Gillespie [9] and Meyer - Nieberg [16]) Let X be a real normed linear space of dimension greater than or equal to three, and let S be a compact operator on X . Then S has a non - trivial invariant subspace .

Proof: The proof for finite - dimensional X is contained in Theorem (1.2.2) . The proof for infinite - dimensional X follows the proof for Corollary (2.1.2) .

Remarks on Theorem (2.1.1) :

(i) The geometrical arguments used in defining S_n and obtaining its properties are taken from Bonsall [3] . They are essentially due to Aronszajn and Smith [1] .

(ii) The dimension argument used to establish that $\liminf_{\nu} F_n \neq (0)$ is due to P. Meyer-Nieberg [16] .

(iii) We give another proof of this theorem for complex X in Chapter IV . This will, however, depend on the Aronszajn-Smith Theorem and Zorn's Lemma .

(iv) Note that we have used the compactness of S in only one place : that is, when obtaining a convergent subsequence of $\{ Sx_n \}_{n=1}^{\infty}$.

(v) Neither Zorn's Lemma nor any consequences of it (eg. the Hahn-Banach Theorem) is used in the proof .

2. The general case.

We come now to the proof of the theorem mentioned previously. For convenience we restate it here .

(2.2.1) Theorem: Let S be a compact operator on an infinite-dimensional real or complex normed linear space X and let $\kappa > 0$. Then S has a non-zero invariant subspace F with $\| S|F \| \leqslant \kappa$.

Proof: We consider separately the real and complex cases. We deal firstly with X complex as it is less involved .

When X is a complex space :--

The operator S must have exactly one of the following properties:

(i) $\ker S \neq (0)$;

(ii) S has no eigen-values ;

(iii) $\ker S = (0)$, $\sigma_0(S)$ is finite and non-empty ;

(iv) $\ker S = (0)$, $\sigma_0(S)$ is infinite .

We consider these four possibilities separately :

(i) Let $F = \ker S$. F has all the desired properties .

(ii) This is simply Theorem (2.1.1) .

(iii) Let $\sigma_0(S) = \{ \lambda_1 , \dots\dots , \lambda_n \}$. For i with $1 \leqslant i \leqslant n$ let m_i be the index of λ_i for S and let

$$M = \bigcap_{i=1}^{n} (S - \lambda_i)^{m_i} X .$$

Since each space $(S - \lambda_i)^{m_i} X$ is of finite codimension in X , M is infinite-dimensional. Also $\sigma_0(S|M) = \emptyset$, so we may apply Theorem (2.1.1) to $S|M$ to obtain the required subspace .

(iv) Theorem (1.3.2) shows that there exists an eigen-value λ of S with $|\lambda| \leqslant \kappa$. Let $F = \ker (S - \lambda)$. F is a non-zero invariant subspace for S and if $x \in F$

$$\| Sx \| = \| \lambda x \| = |\lambda| \, \| x \| \leqslant \kappa \, \| x \| ,$$

so $\| S|F \| \leqslant \kappa$.

When X is a real space :--

Let $\tilde{X}$ be the complexification of X and let $\tilde{S}$ be the complexification of S .

$\tilde{S}$ must satisfy exactly one of the following :

(i) $\ker \tilde{S} \neq (0)$;

(ii) $\widetilde{S}$ has no eigen - values ;

(iii) $\ker \widetilde{S} = (0)$, $\sigma_0(\widetilde{S})$ is finite and non - empty ;

(iv) $\ker \widetilde{S} = (0)$, $\sigma_0(\widetilde{S})$ is infinite .

Once again, we consider these four possibilities separately :

(i) Suppose that $x , y \in X$ are such that $x + iy \in \ker \widetilde{S}$, where not both x and y are zero. Let $F = sp \{ x , y \}$. Then $F \neq (0)$ and F has all the desired properties .

(ii) This case is simply Theorem (2.1.1) . For if L were a non - zero finite - dimensional subspace invariant for S then $\widetilde{S}|\widetilde{L}$ would have an eigen - value .

(iii) Let $\sigma_0(\widetilde{S}) = \{ \lambda_1 , \ldots , \lambda_n \}$. Using Theorem (1.3.7) , for each i , $1 \leqslant i \leqslant n$, we find an $M_i \in \operatorname{Lat} S$ such that $\lambda_i , \lambda_i^* \notin \sigma_0(\widetilde{S}|\widetilde{M}_i)$ and M_i is of finite codimension in X . Let $M = \bigcap_{i=1}^{n} M_i$. Then M is infinite - dimensional and we may apply Theorem (2.1.1) to $S|M$.

(iv) Let $\{ \alpha_n + i\beta_n \}_{n=1}^{\infty}$ be an infinite sequence of distinct eigen - values of $\widetilde{S}$, where $\alpha_n , \beta_n \in \underline{\mathbb{R}}$ for each $n \geqslant 1$. By Theorem (1.3.2) and Lemma (1.3.6) , $\alpha_n , \beta_n \to 0$ as $n \to \infty$. For each $n \geqslant 1$, let $K_n = \ker [\widetilde{S} - (\alpha_n + i\beta_n)]$ and let

$$M_n = \{ x \in X : \text{there exists } y \in X \text{ such that } x + iy \in K_n \} .$$

Clearly $M_n = \operatorname{Re} [K_n + K_n^*]$. Now M_n is a non - zero finite - dimensional invariant subspace of S . We will prove that $\| S|M_n \| \leqslant \kappa$, for some $n \geqslant 1$. We will then have the required result .

Let us suppose that $\| S|M_n \| > \kappa$ for all $n \geqslant 1$. For each

$n \geq 1$, let $x_n \in M_n$ be such that $\| x_n \| = 1$ and $\| Sx_n \| > \kappa$, and let $y_n \in X$ be such that $x_n + iy_n \in K_n$.

Since S is compact, we can find an $\underline{N}$ - subsequence μ such that $z = \lim_{\mu} Sx_n$ exists. Obviously $z \neq 0$.

Now $Sx_n = \alpha_n x_n - \beta_n y_n$ so $z = \lim_{\mu} (\alpha_n x_n - \beta_n y_n)$. But $\| x_n \| = 1$ for all $n \in \mu$, and $\lim_{\mu} \alpha_n = 0$, so $z = - \lim_{\mu} \beta_n y_n$. Thus

$$Sz = -\lim_{\mu} \beta_n Sy_n = -\lim_{\mu} \beta_n (\beta_n x_n + \alpha_n y_n) = -\lim_{\mu} (\beta_n^2 x_n + \alpha_n \beta_n y_n) = 0 .$$

Therefore $\ker S \neq (0)$. This contradicts the assumption .

CHAPTER III

QUESTIONS AND EXAMPLES

In this chapter we generalise a result of Donoghue [6] and we use it to answer certain questions about polynomially compact operators. We give examples of the Main Theorem and we compare it with a theorem of Goldberg [11] .

1. A generalisation of a theorem of Donoghue.

Let H be a separable infinite-dimensional real or complex Hilbert space with orthonormal basis $\{ e_i \}_{i=0}^{\infty}$. Given a bounded sequence of positive real numbers $\{ \alpha_n \}_{n=0}^{\infty}$ we define a weighted shift to be the operator T in $B(H)$ with

$$Te_n = \alpha_n e_{n+1} \quad , \quad n = 0 , 1 , \dots \quad .$$

It is easy to see that if $E_n = \overline{sp} \{ e_n , e_{n+1} , \dots \}$ then $E_n \in \text{Lat } T$ and we may ask what properties the sequence $\{ \alpha_n \}_{n=0}^{\infty}$ should have to ensure that each non-zero element of Lat T is one of the E_n's . Donoghue [6] and Nikolskii (see [15]) have shown that this is the case respectively when $\alpha_n = \frac{1}{2^n}$ and when $\{ \alpha_n \}_{n=0}^{\infty}$ is a non-increasing sequence of strictly positive reals with $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$.

Below in Theorem (3.1.1) we show that the same result follows if the conditions are made somewhat less restrictive. The proof uses methods and syntax taken from Halmos [15, p.97] .

I am grateful to Dr. Alexander for correcting a mistake in the original proof .

(3.1.1) Theorem: Let T be a weighted shift as defined above. Suppose the sequence $\{ \alpha_n \}_{n=0}^{\infty}$ has the property that there exists an integer $m \geq 1$ such that

$$\{ \alpha_{n+m-1} \alpha_{n+m-2} \cdot \cdot \cdot \cdot \cdot \alpha_n \}_{n=0}^{\infty}$$

is a non-increasing sequence, and

$$\sum_{n=0}^{\infty} (\alpha_{n+m-1} \alpha_{n+m-2} \cdot \cdot \cdot \cdot \alpha_n)^2 < \infty ;$$

then

$$\text{Lat } T = \{ E_n \}_{n=0}^{\infty} \cup \{ (0) \} .$$

Proof: Without loss of generality we may assume that $\alpha_n \leq 1$, for all $n \geq 0$.

It is easier to study the invariant subspaces of the adjoint T^* of T . Since $M \in \text{Lat } T$ if and only if $M^{\perp} \in \text{Lat } T^*$ we can deduce the lattice of invariant subspaces of T from that of T^* . So, let $A = T^*$ and for $n \geq 0$ let $M_n = E_n^{\perp}$.

We see that A is defined by the equations :

$$Ae_0 = 0 ,$$

$$Ae_n = \alpha_{n-1} e_{n-1} \quad , \text{ for } n = 1 , 2 , \ldots \quad .$$

Let $M \in \text{Lat } A$. We want to show that M is one of the M_n's or $M = H$.

If $x \in M$ and $x \neq 0$ we define the degree of x , deg x , to be equal to $\sup \{ k : (x, e_k) \neq 0 \}$. This quantity can be infinite .

If $x \in M$ and $\deg x = n < \infty$, then it is easy to see that $x, Ax, A^2x, \ldots\ldots, A^n x$ are linearly independent. But these vectors are in M_{n+1}, which is $(n+1)$-dimensional, so they must span M_{n+1}. Hence $M_{n+1} \subseteq M$.

If $\sup \{ \deg x : x \in M, x \neq 0 \} = n < \infty$, then, from the above result, we see that $M = M_{n+1}$. If M contains vectors of arbitrarily large finite degree then $M_n \subset M$ for infinitely many n, and so $M = H$.

The only remaining case to consider is when M contains a vector x of infinite degree. We shall prove firstly that $e_0 \in M$.

For convenience we define

$$\beta_n = \alpha_{n-1} \alpha_{n-2} \cdots\cdots \alpha_{n-m}$$

for $n \geqslant m$.

If $x = \sum_{r=0}^{\infty} x_r e_r$, then

$$A^n x = \sum_{r=n}^{\infty} x_r \alpha_{r-1} \cdots\cdots \alpha_{r-n} e_{r-n}.$$

If $n \geqslant 1$ is such that $x_n \neq 0$, then

$$(x_n \alpha_{n-1} \cdots\cdots \alpha_0)^{-1} A^n x = e_0 + f_n + g_n$$

where

$$f_n = \sum_{r=n+m}^{\infty} \frac{x_r \alpha_{r-1} \cdots\cdots \alpha_{r-n}}{x_n \alpha_{n-1} \cdots\cdots \alpha_0} e_{r-n}$$

and

$$g_n = \sum_{r=n+1}^{n+m-1} \frac{x_r \alpha_{r-1} \cdots\cdots \alpha_{r-n}}{x_n \alpha_{n-1} \cdots\cdots \alpha_0} e_{r-n}.$$

We will prove that there exists an $\underline{\mathbb{N}}$ - subsequence μ such that

$$\lim_{\mu} f_n = 0 .$$

To do this we will show that, given $\eta > 0$, there exists an $n \geq$ [redacted] 1 with $\| f_n \| < \eta$.

Let $k \geq 2m$ [redacted] be chosen with

$$\sum_{r=k+m}^{\infty} \beta_r^2 < (\alpha_{2m-1} \alpha_{2m-2} \cdots\cdots \alpha_0)^2 \eta^2 ,$$

and then choose $n \geq k$ so that

$$|x_n| = \max \{ |x_r| : r \geq k \} .$$

Let i , j be integers such that $im + j = n$ where [redacted] $1 \leq j \leq m$. [redacted] For $r \geq n+1$ define

$$c_r = \frac{\beta_r \beta_{r-m} \cdots\cdots \beta_{r-(i-1)m}}{\beta_n \beta_{n-m} \cdots\cdots \beta_{n-(i-1)m}} \tag{1}$$

and

$$d_r = \frac{\alpha_{r-im-1} \alpha_{r-im-2} \cdots\cdots \alpha_{r-n}}{\alpha_{j-1} \alpha_{j-2} \cdots\cdots \alpha_0} . \tag{2}$$

Then

$$\| f_n \|^2 = \sum_{r=n+m}^{\infty} \frac{|x_r|^2}{|x_n|^2} \left(\frac{\alpha_{r-1} \alpha_{r-2} \cdots\cdots \alpha_{r-n}}{\alpha_{n-1} \alpha_{n-2} \cdots\cdots \alpha_0} \right)^2$$

$$\leq \sum_{r=n+m}^{\infty} (c_r d_r)^2 .$$

Now $\{ \beta_n \}_{n=m}^{\infty}$ is a decreasing sequence so, for $r \geq n+m$,

$$\beta_{r-m} \leq \beta_n , \quad \beta_{r-2m} \leq \beta_{n-m} , \quad \ldots\ldots , \quad \beta_{r-(i-1)m} \leq \beta_{n-(i-2)m} .$$

Thus

$$c_r \leqslant \frac{\beta_r}{\beta_{n-(i-1)m}} \; .$$

Also

$$\alpha_{r-im-1} \, , \; \alpha_{r-im-2} \, , \; \cdot \; \cdot \; \cdot \; \cdot \; , \; \alpha_{r-n} \leqslant 1 \; ,$$

so

$$d_r \leqslant \frac{1}{\alpha_{j-1} \; \alpha_{j-2} \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \alpha_o} \; .$$

Thus

$$c_r \; d_r \leqslant \frac{\beta_r}{\beta_{n-(i-1)m} \; \alpha_{j-1} \; \alpha_{j-2} \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \alpha_o}$$

$$\leqslant \frac{\beta_r}{\alpha_{2m-1} \; \alpha_{2m-2} \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \alpha_o} \; .$$

Hence

$$\| f_n \|^2 \leqslant \sum_{r=n+m}^{\infty} (c_r \; d_r)^2$$

$$\leqslant \sum_{r=n+m}^{\infty} \frac{\beta_r^{\,2}}{(\alpha_{2m-1} \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \alpha_o)^2}$$

$$\leqslant \eta^2 \; .$$

From this it follows that there exists an $\underline{N}$ - subsequence μ such that $\lim\limits_{\mu} f_n = 0$.

Let $n \in \mu$ and consider g_n . As before let i , j be integers such that $im + j = n$ where [redacted]$1 \leqslant j \leqslant m$, [redacted] and define c_r , d_r by equations (1) and (2) .

Now

$$\| g_n \|^2 = \sum_{r=n+1}^{n+m-1} \frac{|x_r|^2}{|x_n|^2} \left(\frac{\alpha_{r-1} \alpha_{r-2} \cdots\cdots \alpha_{r-n}}{\alpha_{n-1} \alpha_{n-2} \cdots\cdots \alpha_o} \right)^2$$

$$\leqslant \sum_{r=n+1}^{n+m-1} (c_r d_r)^2 .$$

This time we use the inequalities

$$\beta_r \leqslant \beta_n , \quad \beta_{r-m} \leqslant \beta_{n-m} , \ \ldots\ldots , \ \beta_{r-(i-1)m} \leqslant \beta_{n-(i-1)m}$$

which are valid for $r \geqslant n+1$ to deduce that

$$c_r \leqslant 1$$

for $r \geqslant n+1$. Also, as before,

$$d_r \leqslant \frac{1}{\alpha_{j-1} \alpha_{j-2} \cdots\cdots \alpha_o} .$$

Hence

$$\| g_n \|^2 \leqslant \sum_{r=n+1}^{n+m-1} \frac{1}{(\alpha_{j-1} \alpha_{j-2} \cdots\cdots \alpha_o)^2}$$

$$\leqslant (m-1) \frac{1}{(\alpha_{m-1} \alpha_{m-2} \cdots\cdots \alpha_o)^2} .$$

We have thus proved that the sequence $\{ g_n \}_{n \in \mu}$ is bounded .

Since $\{ g_n \}_{n \in \mu}$ is a bounded sequence in the finite-dimensional space $sp \{ e_1 , e_2 , \ldots\ldots , e_{m-1} \}$ there exists an $\underline{\underline{N}}$ - subsequence $\nu \subseteq \mu$ such that $\lim_{\nu} g_n = g$ exists .

Hence $e_o + g \in M$, $e_o + g \neq 0$, and $e_o + g$ has finite degree, so, by previously used arguments, $M_p \subseteq M$ for some p with

$1 \leqslant p \leqslant m$. Thus $e_0 \in M$ and $M_1 \subset M$.

We now prove by induction that $M_n \subset M$ for all $n \geqslant 1$.

Assume that $M_k \subset M$ for some $k \geqslant 1$. Let P_k be the orthogonal projection from H onto E_k . Define $A_k = P_k A | E_k$ and note that A_k satisfies

$$A_k e_k = 0 \ ,$$
$$A_k e_n = \alpha_{n-1} e_{n-1} \ , \quad n = k+1 , k+2 , \ldots .$$

It is easy to see from the induction hypothesis that $P_k M = E_k \cap M$. Now $P_k x$ has infinite degree in E_k so we may apply the same arguments to A_k as we did to A to deduce that $e_k \in E_k \cap M \subset M$. Thus $M_{k+1} \subset M$.

It immediately follows that $M = H$ and so T has the required lattice of invariant subspaces .

Remark: We may regard the above theorem as being about the Hilbert space ℓ_2 . By an obvious adaptation we can obtain a similar result for the Banach spaces ℓ_p where $1 < p < \infty$.

We note the following lemma :

(3.1.2) Lemma: If T satisfies the hypotheses of Theorem (3.1.1) then T is quasi-nilpotent and T^m is compact .

Proof: As before let $\beta_n = \alpha_{n-1} \alpha_{n-2} \ldots \ldots \alpha_{n-m}$ for $n \geqslant m$. Then

$$T^m e_n = \alpha_n \alpha_{n+1} \ldots \ldots \alpha_{n+m-1} e_{n+m}$$
$$= \beta_{n+m} e_{n+m} \ , \quad (n = 0 , 1 , \ldots \ldots) .$$

For $k \geq 0$ let U_k be the operator on H defined by

$$U_k e_n = \begin{cases} \beta_{n+m} \, e_{n+m} , & n = 0 , 1 , \ldots , k \\ 0 , & n = k+1 , k+2 , \ldots . \end{cases}$$

A simple calculation shows that

$$\| T^m - U_k \| = \beta_{m+k+1}$$

and so

$$\| T^m - U_k \| \to 0$$

as $k \to \infty$.

Thus T^m is the norm limit of the sequence $\{ U_k \}_{k=0}^{\infty}$ of finite rank operators and so T^m is a compact operator .

A simple calculation shows that $\sigma_o(T^m) = \emptyset$. Hence T^m and therefore T are quasinilpotent .

Theorem (3.1.1) appears without proof in Nikolskii [17]. We reproduce the statement of Nikolskii's Theorem, which is seemingly far more general than our result, and show that it is in fact easily deducible from it .

(3.1.3) <u>Theorem</u>: Let T be a weighted shift as defined at the beginning of this section. ~~and~~ Suppose there are integers $k , r \geq 1$ and a real number $p > 0$ such that

(i) the sequence

$$\{ \alpha_{n+k-1} \, \alpha_{n+k-2} \, \ldots \ldots \, \alpha_n \}_{n=0}^{\infty}$$

is non- increasing, and

(ii) the series

$$\{ \alpha_{n+r-1} \alpha_{n+r-2} \cdot \cdot \cdot \cdot \cdot \cdot \alpha_n \}_{n=0}^{\infty}$$

is p-summable; that is

$$\sum_{n=0}^{\infty} (\alpha_{n+r-1} \alpha_{n+r-2} \cdot \cdot \cdot \cdot \cdot \alpha_n)^p < \infty .$$

Then

$$\text{Lat } T = \{ E_n \}_{n=0}^{\infty} \cup \{ (0) \} .$$

Proof: First we note the following well-known corollary of the Cauchy-Schwarz inequality : if $\{ x_n \}_{n=0}^{\infty}$ and $\{ y_n \}_{n=0}^{\infty}$ are two p-summable sequences of non-negative reals then $\{ x_n y_n \}_{n=0}^{\infty}$ is $\frac{1}{2}p$-summable.

So, choose a positive integer ι such that $p/2^{\iota} \leq 2$ and let m be a positive integer multiple of k with $m \geq 2^{\iota} r$. Then, clearly,

$$\{ \alpha_{n+m-1} \alpha_{n+m-2} \cdot \cdot \cdot \cdot \cdot \cdot \alpha_n \}_{n=0}^{\infty}$$

is non-increasing and, from the above remark, it can be shown that

$$\sum_{n=0}^{\infty} (\alpha_{n+m-1} \alpha_{n+m-2} \cdot \cdot \cdot \cdot \cdot \alpha_n)^2 < \infty .$$

Thus, with this m, the conditions of Theorem (3.1.1) are satisfied .

2. Examples of the Main Theorem

(3.2.1) Example: Let H be a separable infinite-dimensional real or complex Hilbert space with orthonormal basis $\{ e_i \}_{i=0}^{\infty}$. Let $\{ \alpha_n \}_{n=0}^{\infty}$ be a non-increasing sequence of strictly positive reals with

$$\sum_{n=0}^{\infty} \alpha_n^2 < \infty .$$

Define an operator T on H by the rules

$$Te_n = \alpha_n e_{n+1} \qquad (n = 0 , 1 , \ldots)$$

By Theorem (3.1.1) with $m = 1$ we have that

$$\text{Lat } T = \{ E_n \}_{n=0}^{\infty} \cup \{ (0) \}$$

where $E_n = \overline{\text{sp}} \{ e_n , e_{n+1} , \ldots \}$ for $n = 0 , 1 , \ldots$.

From Lemma (3.1.2) we deduce that T is a compact operator. Then $\| T|E_n \| = \alpha_n$ as is readily seen, and obviously $\alpha_n \to 0$ as $n \to \infty$.

(3.2.2) <u>Example</u>: Let A be the adjoint of the operator T of Example (3.2.1). Let $M_n = E_n^{\perp}$, $n = 0 , 1 , \ldots$. Then

$$\text{Lat } A = \{ M_n \}_{n=0}^{\infty} \cup \{ H \}$$

and $\| A|M_1 \| = 0$ while $\| A|M_n \| = \alpha_0$ for $n \geq 2$.

(3.2.3) <u>Example</u>: Let $H = L_2[0, 1]$ where $L_2[0, 1]$ is the space of (equivalence classes of) complex measurable functions f on the interval $[0, 1]$ such that $\int |f(t)|^2 dt < \infty$ [redacted]. Let $V \in B(H)$ be defined by

$$(Vf)(x) = \int_0^x f(t) \, dt , \qquad x \in [0, 1] .$$

V is called a Volterra operator. The following information about V may be found in Halmos [15] :

(i) V is compact and quasinilpotent ;

(ii) $\text{Lat } V = \{ M_\alpha \}_{\alpha \in [0,1]}$ where

$$M_\alpha = \{ f \in H : f(t) = 0 \text{ a.e. for } t \in [0, \alpha] \}$$

for each $\alpha \in [0, 1]$;

(iii) $\| V \| = 2\pi^{-1}$.

The proofs that V has properties (i) and (ii) may be found in [6] . Property (iii) is proved in [15] .

We will show that

$$\| V|M_\alpha \| = 2(1 - \alpha)\pi^{-1} .$$

It will follow that $\| V|M_\alpha \| \to 0$ as $\alpha \to 1$.

The case $\alpha = 1$ is trivial, so fix α with $0 \leq \alpha < 1$. Define the map $\theta : H \to M_\alpha$ by

$$(\theta f)(x) = \begin{cases} 0 & x \in [0, \alpha] \\ (1 - \alpha)^{-\frac{1}{2}} f\left(\frac{x - \alpha}{1 - \alpha} \right) & x \in (\alpha, 1] \end{cases}$$

It is easy to see that θ is a surjective isometry and that

$$V(\theta f) = (1 - \alpha)\theta(Vf) .$$

From this follows the required result .

3. Polynomially compact operators.

(3.3.1) Definition: Let X be a normed linear space. An operator T in $B(X)$ is said to be polynomially compact if there is a non-zero polynomial p such that $p(T)$ is compact .

Bernstein and Robinson [2] and Bonsall [3] have proved that polynomially compact operators on infinite-dimensional spaces have non-trivial invariant subspaces. We may thus ask if a similar theorem to the Main Theorem holds for polynomially compact operators. We ask two questions, the first of which is very naive .

(3.3.2) Question: In Theorem (2.2.1) can we replace the condition that S is compact by the condition that S is polynomially compact ?

The answer no is readily found. Take the identity on X . This is polynomially compact (under the polynomial $x \mapsto x-1$) but the norm of the restriction to any non - zero invariant subspace is 1 .

(3.3.3) Question: In Theorem (2.2.1) can we replace the condition that S is compact by the condition that S is polynomially compact and quasinilpotent ?

The answer is again no . We give a counter - example based on Theorem (3.1.1) .

Let H be a separable infinite - dimensional Hilbert space with orthonormal basis $\{ e_n \}_{n=0}^{\infty}$. Define T in $B(H)$ by

$$Te_n = \begin{cases} e_{n+1} & , \ n \ \text{even} \ , \ n \geq 0 \\ \dfrac{e_{n+1}}{n} & , \ n \ \text{odd} \ , \ n \geq 1 \ . \end{cases}$$

The operator T satisfies the hypotheses of Theorem (3.1.1) with $m = 2$ so, if $M \in \text{Lat } T$, $M \neq (0)$, then

$$M = \overline{\text{sp}} \{ e_n , e_{n+1} , \ldots\ldots \}$$

for some $n \geq 0$. Either $\| Te_n \| = 1$ or $\| Te_{n+1} \| = 1$ so $\| T|M \| \geq 1$. But $\| T \| = 1$ so $\| T|M \| = 1$. Lemma (3.1.2) shows that T^2 is compact and T is quasinilpotent .

4. A theorem of Goldberg.

It is of interest to compare Theorem (2.2.1) with Theorem III.23 of Golberg [11] . We reproduce it here in the form it takes for operators on a Banach space .

(3.4.1) Theorem: Let S be an operator on a Banach space X . S is a compact operator if and only if, for every $\kappa > 0$ there exists a subspace N having finite codimension in X such that $\| S|N \| \leq \kappa$.

Remark: N need not be invariant for S .

The following question springs to mind :

(3.4.2) Question: Let S be a compact operator on a complex Banach space X . Given $\kappa > 0$, does there exist a subspace $M \in \text{Lat } S$ having finite codimension in X such that $\| S|M \| \leq \kappa$?

Once again the answer is, in general, no .

If an operator has an invariant subspace of non - zero finite codimension then its adjoint has an eigen - value. We construct a compact operator whose adjoint has no eigen - values. This will be the required counter - example .

Let H be a complex Hilbert space with orthonormal basis $\{e_n\}_{n=-\infty}^{\infty}$. Define $S \in B(H)$ by the equations

$$Se_n = \frac{e_{n+1}}{|n|+1} \quad , (n \in \underline{Z}) .$$

Clearly, S is a compact operator .

It is easy to check that S^* has no eigen-values and so, the only invariant subspace of S , of finite codimension in H , is H itself. But $\| S|H \| = 1$. Hence there is no subspace $M \in \mathrm{Lat}\, S$ of finite codimension in X satisfying $\| S|M \| < 1$.

CHAPTER IV

TWO GENERALISATIONS OF THE MAIN THEOREM

In this chapter we generalise the Main Theorem in two directions. The first involves sets of operators and it is used to stregthen results of Bernstein and Robinson, Feldman, and others. We also give an alternative proof of the Main Theorem. For the second direction we consider compact operators on locally convex topological vector spaces. For both these results we need the algebraic results of Section 1 .

Suppose we have a class of operators Λ on a normed linear space X for which there is a theorem which states that $\operatorname{Lat} \Lambda$ contains a non-trivial subspace. It may be that scrutiny of the theorem reveals that Λ has the property that if $M \in \operatorname{Lat} \Lambda$ and $\dim M \geqslant 2$ then $\operatorname{Lat} (\Lambda|M)$ contains a non-trivial subspace. Naturally, we are defining $\Lambda|M = \{ A|M : A \in \Lambda \}$.

Many invariant subspace theorems turn out to have this property. For an obvious example, take $\Lambda = \{S\}$, where S is a compact operator. The Aronszajn - Smith Theorem [1] (see Corollary (2.1.2)) states that (for X a complex normed linear space with $\dim X \geqslant 2$) S has a non-trivial invariant subspace. Since the restriction of S to any invariant subspace is again a compact operator we may use the Aronszajn - Smith Theorem to deduce that $\operatorname{Lat} \Lambda = \operatorname{Lat} S$ has the properties described above.

We isolate this property of subspaces in Definition (4.1.1) .

1. Some algebraic results.

This section is concerned with proving two lemmas in linear algebra. We shall use such words as 'operator' and 'subspace' but these will be taken to have purely algebraic meaning ; their topological connotations are ignored .

(4.1.1) Definition: Let X be a complex linear space. A set $\mathcal{L}$ of subspaces of X is said to be hereditary if :

(i) $X \in \mathcal{L}$;

(ii) $\mathcal{L}$ is closed under arbitrary intersection ;

(iii) for each $M \in \mathcal{L}$ with $\dim M \geq 2$ there exists an $N \in \mathcal{L}$ such that $(0) \neq N \subset M$.

Remark: Though we could consider real spaces in the above definition, we prefer for aesthetic reasons to restrict the definition to complex spaces .

To explain our motives for this, note that Theorem (1.2.1) shows that the set of invariant subspaces of a linear operator on a finite-dimensional complex vector space is hereditary. An example of an operator on a real space whose lattice of invariant subspaces is not hereditary is given by the operator defined on $\mathbb{R}^2$ with the usual basis by the matrix

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

It is a simple consequence of Zorn's Lemma that if $\mathcal{L}$ is a set of subspaces of X then, given any chain $\mathcal{C}$ contained in $\mathcal{L}$, there exists a chain $\mathcal{M} \supseteq \mathcal{C}$ which is maximal in $\mathcal{L}$.

By the phrase '$\mathfrak{M}$ is maximal in $\mathcal{L}$' we mean that $\mathfrak{M}$ has the property that if $\mathfrak{N}$ is a chain satisfying $\mathfrak{M} \subseteq \mathfrak{N} \subseteq \mathcal{L}$ then $\mathfrak{M} = \mathfrak{N}$. We shall reserve the word 'maximal' for this concept.

We note that if $\mathcal{L}$ is an hereditary set of subspaces and $\mathfrak{M}$ is a maximal chain contained in $\mathcal{L}$, then $\mathfrak{M}$ is an hereditary set of subspaces .

(4.1.2) Definition: Let $\mathfrak{N}$ be a set of subspaces of X . We define

$$\mathfrak{N}_0 = \{ M \in \mathfrak{N} : M \neq (0) \} .$$

(4.1.3) Lemma: Let $\mathcal{L}$ be an hereditary set of subspaces of a complex linear space X and let $\mathfrak{M}$ be a maximal chain in $\mathcal{L}$. Then $\dim (\cap \mathfrak{M}_0) \leq 1$.

Proof: If $\dim (\cap \mathfrak{M}_0) \geq 2$ then there exists an $N \in \mathcal{L}$ such that $(0) \neq N \subset (\cap \mathfrak{M}_0)$. Obviously $\mathfrak{M} \cup \{N\}$ is a chain in $\mathcal{L}$ strictly containing $\mathfrak{M}$. This contradicts the maximality of $\mathfrak{M}$.

(4.1.4) Lemma: Let X be an infinite - dimensional complex linear space and let Σ be a non - empty set of operators on X . Suppose $\mathcal{L}$ is an hereditary set of subspaces of X with the following two properties :

(i) $SM \subseteq M$, ($S \in \Sigma$, $M \in \mathcal{L}$) ;

(ii) for each $S \in \Sigma$ and each $\lambda \in \underline{C}_0$, there exists an $N \in \mathcal{L}$ such that N is of finite codimension in X and λ is not an eigen - value of $S|N$.

Then, <u>either</u> there is an $L \in \mathcal{L}$ with $\dim L = 1$ and $L \subseteq \bigcap_{S\in\Sigma} \ker S$, <u>or</u> there exists a chain $\mathfrak{M}$ in $\mathcal{L}$ with $\cap\mathfrak{M}_0 = (0)$.

<u>Proof</u>: Let I be an indexing set for $\Sigma \times \underline{C}_0$ and let $\leqslant$ be a well-ordering of I . (See [13] for the relevant properties of well-ordering) . For each $\iota \in I$ let $N_\iota \in \mathcal{L}$ be such that N_ι is of finite codimension in X and λ_ι is not an eigen-value of $S_\iota | N_\iota$. Let

$$P = \bigcap_{\iota\in I} N_\iota$$

and for each $j \in I$ let

$$R_j = \bigcap_{\iota\leqslant j} N_\iota \quad .$$

There are three possibilities :

(a) $P \neq (0)$;

(b) $P = (0)$ and R_j is infinite-dimensional for each $j \in I$;

(c) $P = (0)$ and R_j is finite-dimensional for some $j \in I$.

(a) Let $\mathfrak{M}$ be a maximal chain in $\mathcal{L}$ with $P \in \mathfrak{M}$. By Lemma (4.1.3) , $\dim (\cap\mathfrak{M}_0) \leqslant 1$. If $\dim (\cap\mathfrak{M}_0) = 1$ let

$$L = \cap\mathfrak{M}_0 \quad .$$

To each $S \in \Sigma$ there is a $\lambda_S \in \underline{C}$ such that

$$S|L = \lambda_S I|L \quad .$$

But, by the construction of P and the fact that $L \subseteq P$, it follows that $\lambda_S = 0$ for all $S \in \Sigma$. Thus $L \subseteq \bigcap_{S\in\Sigma} \ker S$.

(b) Let $\mathfrak{M} = \{ R_j \}_{j \in I}$; then $\cap \mathfrak{M}_o = (0)$.

(c) Let

$$j_o = \inf \{ j \in I : R_j \text{ is finite-dimensional} \}$$

and let

$$K = \bigcap_{j < j_o} R_j \, .$$

If $K = (0)$ then the chain $\mathfrak{M} = \{ R_j \}_{j < j_o}$ has the property that

$$\cap \mathfrak{M}_o = (0) \, .$$

If $K \neq (0)$ then, since $R_{j_o} = K \cap N_{j_o}$ is finite-dimensional and N_{j_o} is of finite codimension in X , it follows that K is finite-dimensional .

Since $K \cap [\bigcap_{j \geq j_o} N_j] = (0)$ and K is finite-dimensional, there exists an integer n and $j_1 , \ldots , j_n \in I$ such that

$$K \cap [\bigcap_{k=1}^{n} N_{j_k}] = (0) \, .$$

Let $Q = \bigcap_{k=1}^{n} N_{j_k}$ and note that Q has finite codimension in X . For each $j \in I$ with $j < j_o$, let $R_j^o = R_j \cap Q$. Since R_j is infinite-dimensional for $j < j_o$ and Q is of finite codimension in X , R_j^o is infinite-dimensional . Also

$$\bigcap_{j < j_o} R_j^o = Q \cap K = (0)$$

and so the chain $\mathfrak{M} = \{ R_j^o \}_{j < j_o}$ satisfies $\cap \mathfrak{M}_o = (0)$.

We may vary the conditions of Lemma (4.1.4) and obtain much the same result .

For the rest of this section we consider the real and complex cases together, as the definitions apply naturally to both cases .

(4.1.5) Definition: Let X be a linear space. A set $\mathcal{L}$ of subspaces of X is said to be quasi-herditary if :

(i) $X \in \mathcal{L}$;

(ii) $\mathcal{L}$ is closed under arbitrary intersection ;

(iii) for each $M \in \mathcal{L}$ with M of infinite dimension, there exists an $N \in \mathcal{L}$ such that $(0) \neq N \subset M$.

(4.1.6) Lemma: Let $\mathcal{L}$ be a quasi-hereditary set of subspaces of a linear space X and let $\mathfrak{M}$ be a maximal chain in $\mathcal{L}$. Then $\dim (\cap \mathfrak{M}_0) < \infty$.

Proof: This follows the same lines as the proof of Lemma (4.1.3) .

(4.1.7) Lemma: Let X be an infinite-dimensional complex [respectively, real] linear space and let Σ be a non-empty set of operators on X . Suppose $\mathcal{L}$ is a quasi-hereditary set of subspaces of X with the following properties :

(i) $SM \subseteq M$, $(S \in \Sigma , M \in \mathcal{L})$;

(ii) for each $S \in \Sigma$ and each $\lambda \in \underline{C}_0$, there exists an $N \in \mathcal{L}$ such that N is of finite codimension in X and λ is not an eigen-value of $S|N$ [respectively, of $\widetilde{S}|\widetilde{N}$, where $\widetilde{S}$, $\widetilde{N}$ denote the complexifications of S , N] ;

(iii) $\ker S \in \mathcal{L}$ for all $S \in \Sigma$.

Then, <u>either</u> there is an $L \in \mathcal{L}_0$ with $\dim L < \infty$ and $L \subseteq \bigcap_{S\in\Sigma} \ker S$, <u>or</u> there is a chain $\mathfrak{M}$ in $\mathcal{L}$ such that $\cap\mathfrak{M}_0 = (0)$.

<u>Proof</u>: Let I , $\leqslant$, N_i ($i \in I$) , P , R_j ($j \in I$) be as in the proof of Lemma (4.1.4) .

There are three possibilities :

(a) $P \neq (0)$;

(b) $P = (0)$ and R_j is infinite - dimensional for each $j \in I$;

(c) $P = (0)$ and R_j is finite - dimensional for some $j \in I$.

(a) Let $\mathfrak{M}$ be a maximal chain in $\mathcal{L}$ with $P \in \mathfrak{M}$. By Lemma (4.1.6) , $\dim (\cap\mathfrak{M}_0) < \infty$. If $\cap\mathfrak{M}_0 = (0)$ then $\mathfrak{M}$ has the required properties. If $\cap\mathfrak{M}_0 \neq (0)$ let $L = \cap\mathfrak{M}_0$. Since L is finite - dimensional, to each $S \in \Sigma$ there is a $\lambda_S \in \underline{C}$ such that λ_S is an eigen - value of $S|L$ [respectively, of $\tilde{S}|\tilde{L}$ where $\tilde{L}$ is the complexification of L] . By the construction of P , $\lambda_S = 0$ for each $S \in \Sigma$.

If there is an $S_0 \in \Sigma$ such that $L \nsubseteq \ker S_0$, then $L \cap \ker S_0 \in \mathcal{L}$ is non - zero and is strictly contained in L . This contradicts the maximality of $\mathfrak{M}$. Thus $L \subseteq \bigcap_{S\in\Sigma} \ker S$.

(b) Let $\mathfrak{M} = \{ R_j \}_{j\in I}$, then $\cap\mathfrak{M}_0 = (0)$.

(c) This follows the same lines as (c) of Lemma (4.1.4) .

(4.1.8) Example: Let T and A be the operators of Examples (3.2.1) and (3.2.2) respectively. Then Lat T and Lat A are hereditary sets of subspaces. If $\lambda \in \underline{C}_0$, then H is of finite codimension in H and λ is not an eigen-value of $T|H$ or $A|H$.

Thus the conditions of Lemma (4.1.4) are satisfied when :

(i) $\Sigma = \{T\}$, $\mathcal{L} = \text{Lat } T$,

and when

(ii) $\Sigma = \{A\}$, $\mathcal{L} = \text{Lat } A$.

For (i) the chain $\mathfrak{M} = \mathcal{L}$ satisfies $\cap \mathfrak{M}_0 = (0)$, while $\ker T = (0)$. For (ii) $\dim(\ker A) = 1$ and there is no chain $\mathfrak{M} \subseteq \mathcal{L}$ with $\cap \mathfrak{M}_0 = (0)$. We thus see that the alternatives in the conclusion of Lemma (4.1.4) are essential .

2. A generalisation to sets of operators.

We now return to topological considerations. The following lemma is the means whereby we apply the two lemmas of Section 1 to obtain topological results .

(4.2.1) Lemma: Let X be a real or complex infinite-dimensional normed linear space and let S be a compact operator on X . Suppose $\mathfrak{M}$ is a chain contained in Lat S with $\cap \mathfrak{M}_0 = (0)$. Then

$$\inf \{ \| S|M \| : M \in \mathfrak{M}_0 \} = 0 .$$

Proof: Suppose that

$$\inf \{ \| S|M \| : M \in \mathfrak{M}_0 \} > 0$$

and let $\kappa > 0$ be such that

$$\kappa < \inf \{ \| S|M \| : M \in \mathfrak{M}_0 \} .$$

For each $M \in \mathfrak{M}_0$ let

$$P_M = \text{cl} \{ Sx : x \in M , \| x \| \leqslant 1 , \| Sx \| \geqslant \kappa \} .$$

Note that $\emptyset \neq P_M \subset M$ for all $M \in \mathfrak{M}_0$.

Since P_M is compact for each $M \in \mathfrak{M}_0$, it follows that if $\bigcap_{M \in \mathfrak{M}_0} P_M = \emptyset$ then we can find an integer n and subspaces $M_1 , M_2 , \ldots , M_n \in \mathfrak{M}_0$ with $\bigcap_{i=1}^{n} P_{M_i} = \emptyset$. But this implies that, if $M_0 = \bigcap_{i=1}^{n} M_i$, then $P_{M_0} = \emptyset$, which is impossible. Thus $\bigcap_{M \in \mathfrak{M}_0} P_M \neq \emptyset$ and, since $0 \notin P_M$ for each M , $\bigcap_{M \in \mathfrak{M}_0} M \neq (0)$.

(4.2.2) <u>Theorem</u>: Let X be an infinite-dimensional real or complex normed linear space and let Λ be a subset of $B(X)$ such that Lat Λ is a quasi-hereditary set of subspaces of X . Suppose Σ is a non-empty subset of Λ consisting of compact operators with the property that $SA = AS$ for all $S \in \Sigma$ and all $A \in \Lambda$. Then, <u>either</u> there is a chain $\mathcal{C}$ of infinite-dimensional subspaces in Lat Λ with $\bigcap \mathcal{C}_0 = (0)$ and

$$\inf \{ \| S|M \| : M \in \mathcal{C}_0 \} = 0$$

for all $S \in \Sigma$ <u>or</u>

$$\bigcap_{S \in \Sigma} \ker S \neq (0) .$$

<u>Proof</u>: Let $\mathcal{L} = \text{Lat } \Lambda$. We will show that $\mathcal{L}$ and Σ satisfy the hypotheses (i) , (ii) , and (iii) of Lemma (4.1.7) .

(i) This is automatic.

(ii) For X a complex space :--

Let $S \in \Sigma$, $\lambda \in \underline{C}_0$. Let m be the index of λ for S and let $N = (S - \lambda)^m X$. If $A \in \Lambda$ then

$$AN = A(S - \lambda)^m X = (S - \lambda)^m AX \subseteq (S - \lambda)^m X = N \ .$$

Also it follows from Theorem (1.3.5) that N is a subspace of finite codimension in X and λ is not an eigen-value of $S|N$. Thus $N \in \text{Lat } \Lambda$ has the required properties .

For X a real space :--

Let $S \in \Sigma$, $\lambda \in \underline{C}_0$. Let m be the index of λ for $\tilde{S}$ and let

$$N = \text{Re } [\ (\tilde{S} - \lambda)^m \tilde{X} \cap (\tilde{S} - \lambda^*)^m \tilde{X} \ ] \ .$$

A similar argument to that above shows that $N \in \text{Lat } \Lambda$. Moreover λ is not an eigen-value of $\tilde{S}|\tilde{N}$ and N is of finite codimension in X .

(iii) Evidently, $\ker S \in \text{Lat } \Lambda$ for each $S \in \Sigma$.

It follows now that <u>either</u> $\bigcap_{S \in \Sigma} \ker S \neq (0)$ <u>or</u> there is a chain $\mathcal{C}$ in $\text{Lat } \Lambda$ such that $\cap \mathcal{C}_0 = (0)$. Clearly, it would be impossible for $\mathcal{C}_0$ to contain any finite-dimensional spaces .

Using Lemma (4.2.1) , we see that $\mathcal{C}$ has the property that

$$\inf \{ \ \| S|M \| : M \in \mathcal{C}_0 \ \} = 0$$

for all $S \in \Sigma$.

As promised in Chapter II we give an alternative proof of the Main Theorem. The result we give below is stronger than the Main Theorem but it should be noted that an adaptation of the methods in Chapter II will yield this stronger result without using Zorn's Lemma or any of its consequences .

(4.2.3) Corollary: Let S be a compact operator on a real or complex infinite - dimensional normed linear space X . Then, if $\ker S = (0)$, there exists a chain $\mathcal{C}$ of infinite - dimensional subspaces in Lat S with $\cap \mathcal{C}_0 = (0)$ and

$$\inf \{ \| S|M \| : M \in \mathcal{C}_0 \} = 0 .$$

Proof: Let $\Lambda = \Sigma = \{S\}$. We use the Aronszajn - Smith Theorem [1] (Corollary (2.1.2)) for complex spaces and the results of Gillespie [9] or Meyer - Nieberg [16] (Corollary (2.1.3)) for real spaces, to deduce that Lat Λ is quasi - hereditary .

Thus Λ satisfies the hypotheses of Theorem (4.2.2) and the required result follows immediately .

On the same lines we now strengthen various other invariant subspace results .

In all of the following theorems and definitions X is an infinite - dimensional normed linear space and S , T are operators on X with S compact and $ST = TS$.

(4.2.4) <u>Theorem</u>: If $p(T) = S$ for some non-trivial polynomial p then there is a non-trivial subspace invariant for both S and T.

<u>Proof</u>: For the complex case in Hilbert space see [2] or [14]. For the complex case for normed linear spaces see [3]. For the real case see [9] or [16].

<u>Definition</u>: Let X be a separable complex Hilbert space and let $T \in B(X)$. We say that T is <u>quasi-triangular</u> if there exists a sequence of orthogonal finite-rank projections $\{ P_n \}_{n=1}^{\infty}$ such that

$$\| P_n x - x \| \to 0$$

as $n \to \infty$ for each $x \in X$ and

$$\| P_n T P_n - T P_n \| \to 0$$

as $n \to \infty$.

(4.2.5) <u>Theorem</u>: Let X be a separable complex Hilbert space and suppose that T is a quasi-triangular operator. Suppose, in addition, that there is a sequence of polynomials in T, $\{ p_m(T) \}$, such that

$$\| p_m(T) - S \| \to 0$$

as $m \to \infty$.

If $S \neq 0$, then there is a non-trivial subspace invariant for both S and T.

<u>Proof</u>: See Arveson and Feldman [8].

Definition: A rational function in the operator T is an operator of the form $p(T)[q(T)]^{-1}$ where p , q are polynomials and $[q(T)]^{-1}$ exists .

(4.2.6) Theorem: Let X be a separable complex Hilbert space and let T be a quasi-triangular operator. Suppose that S is in the uniform closure of the set of rational functions in T .

If $S \neq 0$, then there is a non-trivial subspace invariant for both S and T .

Proof: See Pearcy and Salinas [18] .

(4.2.7) Theorem: Let X be a separable complex Hilbert space and let T be a quasi-triangular operator. Suppose that there is a sequence of polynomials $\{ p_n \}$ and a sequence of operators $\{ S_n \}$ such that

$$\lim (S_n x,y) = (Sx,y)$$

for all $x , y \in X$ and

$$\lim \| S_n - p_n(T) \| = 0 .$$

If $S \neq 0$ then there exists a non-trivial subspace invariant for both S and T .

Proof: See Deckard , Douglas , and Pearcy [5] .

(4.2.8) Theorem: Let X be a normed linear space and let T be quasi-nilpotent. Suppose there exists a sequence of polynomials $\{ p_n \}$ such that

$$\| p_n(T) - S \| \to 0 .$$

If $S \neq 0$ then there exists a non-trivial subspace invariant for both S and T.

Proof: The proofs for both the real and complex cases appears in [9] but the complex case has been published in [10].

Definition: Let T be an operator on a complex Banach space X and let $\{ a_n \}_{n=0}^{\infty}$ be a sequence of complex numbers. If $\sum_{n=0}^{\infty} a_n T^n$ converges in norm we call the sum a basic analytic expression in T.

A general analytic expression in T is an operator of the form $p(A_1 , A_2 , \ldots , A_n)$ where p is a polynomial and $A_1 , A_2 , \ldots , A_n$ are basic analytic expressions in T.

(4.2.9) Theorem: Let X be a complex Banach space and suppose some general analytic expression in T is equal to S.

If $S \neq 0$ then there exists a non-trivial subspace invariant for both S and T.

Proof: See [4].

A quasi-nilpotent operator with a cyclic vector on a Hilbert space is quasi-triangular [8]. Hence Theorems (4.2.5,6,7) are true with quasi-triangular replaced by "quasi-nilpotent".

We now strengthen the conclusion of all these theorems in the following way :

(4.2.10) Corollary: Let X , S , T satisfy the hypotheses of one of Theorems (4.2.4,5,6,7,8,9) with "quasi-triangular" replaced by "quasi-nilpotent" in (4.2.5,6,7). Then if $\ker S = (0)$ there exists a chain $\mathcal{C}$ in $\operatorname{Lat}\{ S , T \}$ such that $\cap \mathcal{C}_0 = (0)$ and

$$\inf \{ \| S|M \| : M \in \mathcal{C}_0 \} = 0 .$$

Remark: For the proof of this corollary we obviously do not need the requirement of some of the theorems that $S \neq 0$.

Proof: Clearly if $\ker S = (0)$ it follows from the specified theorem that $\mathrm{Lat}\{S, T\}$ is a quasi-hereditary set of subspaces.* Thus with $\Sigma = \{S\}$ and $\Lambda = \{S, T\}$ the conditions of Theorem (4.2.2) are satisfied. Hence the required result is obtained .

Many invariant subspace results have been conjectured. Probably, all of those that involve compact operators can be strengthened by Theorem (4.2.2) to give information about the norm. As an example, consider the following question which appears in [3 p. 89] .

Question: Let X be a complex normed linear space with $\dim X \geq 2$ and let S_1, S_2 be compact operators on X with $S_1 S_2 = S_2 S_1$. Do S_1 and S_2 have a non-trivial invariant subspace in common ?

If the answer to this question is yes, in general (for finite-dimensional spaces this is well-known) then it follows from Theorem (4.2.2) that the answer to the following question is also yes .

Question: Let X be a complex infinite-dimensional normed linear space and let S_1 and S_2 be compact operators on X with $S_1 S_2 = S_2 S_1$. Given $\eta_1 > 0$ and $\eta_2 > 0$, is there a non-zero subspace F invariant for both S_1 and S_2 with $\| S_1 | F \| \leq \eta_1$ and $\| S_2 | F \| \leq \eta_2$?

* In the quasi-nilpotent cases this uses the fact that the restriction of a quasi-nilpotent operator to an invariant subspace is also quasi-nilpotent. The corresponding result for quasi-triangular operators is not true in general .

3. Locally convex topological vector spaces.

In [3] Bonsall shows that compact operators on locally convex topological vector spaces (for definitions, see below) have non-trivial invariant subspaces. We strengthen this result in a similar fashion to the way we have strengthened results in Section 2 . We will only discuss the complex case : the real case follows by analogy with Section 2 .

We take Robertson and Robertson [20] as our source for definitions and results about topological vector spaces. Note that they use (as we shall use) 'convex space' as short-hand for 'locally convex topological vector space over the field of complex numbers'. Also, 'separated' is a synonym for 'Hausdorff'. Subspaces are closed in the given topology .

We first take the necessary theorem from [20] and then we reproduce Bonsall's result in the form we require.

(4.3.1) Definition: Let X be a topological vector space. A linear map from X to itself is said to be a compact operator if there is a neighbourhood N of the origin such that TN is contained in a compact set .

(4.3.2) Theorem: [20 p. 148] Let T be a compact operator on a separated convex space X . If $\lambda \in \underline{C}_0$ then there is a subspace M invariant for T and of finite codimension in X such that λ is not an eigen-value of $T|M$.

(4.3.3) <u>Theorem</u>: [3 p. 77] Let X be a separated convex space of dimension greater than or equal to two and let T be a compact operator on X. Then T has a non-trivial invariant subspace.

<u>Proof</u>: Let N be a neighbourhood of the origin with $\overline{TN}$ compact. Let C be an absolutely convex neighbourhood of the origin contained in N and let $|\cdot|$ be the semi-norm associated with C. The set

$$\text{cl}\ \{ Tx : |x| \leqslant 1 \}$$

is compact and so the continuous function $|\cdot|$ is bounded on it. Hence there is a constant $A \geqslant 0$ such that

$$|Tx| \leqslant A|x|$$

for all $x \in X$.

Let

$$P = \{ x \in X : |x| = 0 \} .$$

P is clearly a subspace invariant for T and so the theorem is proved unless $P = (0)$ or X.

If $P = X$ then TX is contained in a compact set. TX must equal (0), so $T = 0$. Any one-dimensional subspace is invariant.

If $P = (0)$ then $|\cdot|$ is a continuous norm on X. The norm topology τ_n is weaker than the given topology τ, hence, T is a compact operator on $(X , |\cdot|)$. Using Corollary (2.1.2) we can find a non-trivial subspace L invariant for T (closure of subspace is in τ_n). But τ_n is weaker than τ so L is closed in τ.

Using these two theorems together with Lemma (4.1.4) and employing ideas similar to Lemma (4.2.1) we obtain the following :

(4.3.4) Theorem: Let T be a linear map from a separated convex space X to itself with the property that $\overline{TN}$ is compact for some neighbourhood N of the origin. Then, given an arbitrary neighbourhood K of the origin, there exists a non-zero invariant subspace M of T with $T(N \cap M) \subseteq K$.

Proof: If $\ker T \neq (0)$ the result is established by taking $M = \ker T$. So, assume $\ker T = (0)$.

It follows easily from Theorem (4.3.3) that Lat T is an hereditary set of subspaces. This, together with Theorem (4.3.2), ensures that the conditions of Lemma (4.1.4) are satisfied with $\Sigma = \{ T \}$ and $\mathcal{L} = \text{Lat } T$. Hence, there is a chain $\mathfrak{M}$ contained in Lat T with $\cap \mathfrak{M}_0 = (0)$.

Let us suppose that $T(N \cap M) \not\subseteq K$ for each $M \in \mathfrak{M}_0$. This means that if D is the complement of K in X then

$$TN \cap TM \cap D \neq \emptyset .$$

This implies that

$$\overline{TN} \cap TM \cap \overline{D} \neq \emptyset ,$$

which in turn implies that

$$\overline{TN} \cap M \cap \overline{D} \neq \emptyset .$$

But $\mathfrak{M}_0$ is a chain and $\overline{TN} \cap \overline{D}$ is a compact set, so

$$\overline{TN} \cap (\cap \mathfrak{M}_0) \cap \overline{D} \neq \emptyset .$$

Thus $\overline{D} \cap (\cap \mathfrak{M}_0) \neq \emptyset$, which is a contradiction since $0 \notin \overline{D}$. Hence, there is an $M \in \mathfrak{M}_0$ such that $T(N \cap M) \subseteq K$.

CHAPTER V

CONTINUITY OF THE NORM

In this chapter we introduce the notion of continuity of the norm of an operator when restricted to a chain of subspaces. We discuss this with particular reference to compact operators and give examples which answer questions which arise naturally from the theorems .

1. Definitions and elementary results.

(5.1.1) Definition: Let X be a normed linear space. A chain $\mathcal{C}$ of subspaces of X is said to be complete if (0) , $X \in \mathcal{C}$ and the subspaces $\cap \mathcal{C}_1$ and $\mathrm{cl}\,(\cup \mathcal{C}_1)$ are in $\mathcal{C}$ for each non - empty subset $\mathcal{C}_1$ of $\mathcal{C}$.

(5.1.2) Definition: Let $\mathcal{C}$ be a complete chain of subspaces of a normed linear space X and let $M \in \mathcal{C}$. We define the subsets $[\mathcal{C}, M]_+$ and $[\mathcal{C}, M]_-$ of $\mathcal{C}$ as follows :

$$[\mathcal{C}, M]_+ = \{ N \in \mathcal{C} : N \supset M \} ,$$

$$[\mathcal{C}, M]_- = \{ N \in \mathcal{C} : N \subset M \} .$$

(5.1.3) Definition: Let $\mathcal{C}$ be a complete chain of subspaces of a normed linear space X and let $M \in \mathcal{C}$. We say that M is an upper limit point of $\mathcal{C}$ if $[\mathcal{C}, M]_+ \neq \emptyset$ and

$$M = \cap [\mathcal{C}, M]_+ .$$

Similarly , we say that M is a lower limit point of $\mathcal{C}$ if $[\mathcal{C}, M]_- \neq \phi$ and ,

$$M = \mathrm{cl} \cup [\ \mathcal{C}\ ,\ M\ ]_-\ .$$

(5.1.4) Definition: Let $\mathcal{C}$ be a chain of subspaces of a normed linear space X and let f be a map from $\mathcal{C}$ to $\underline{R}$. The map f is said to be order - preserving if M , N $\in \mathcal{C}$, $M \subseteq N$ implies that $f(M) \leq f(N)$.

(5.1.5) Definition: Let $\mathcal{C}$ be a complete chain of subspaces of a normed linear space X and let M be an upper limit point of $\mathcal{C}$. Let f be an order - preserving map from $\mathcal{C}$ to $\underline{R}$. We say that f is upper semi - continuous at M if

$$f(M) = \inf \{\ f(N)\ :\ N \in [\ \mathcal{C}\ ,\ M\ ]_+\ \}\ .$$

Similarly , when instead, M is a lower limit point of $\mathcal{C}$ we say that f is lower semi - continuous at M if

$$f(M) = \sup \{\ f(N)\ :\ N \in [\ \mathcal{C}\ ,\ M\ ]_-\ \}\ .$$

(5.1.6) Definition: Let $\mathcal{C}$ be a complete chain of subspaces of a normed linear space X and let f be an order - preserving map from $\mathcal{C}$ to $\underline{R}$. We say that f is upper semi - continuous on $\mathcal{C}$ if f is upper semi - continuous at each upper limit point of $\mathcal{C}$.

Similarly , we say that f is lower semi - continuous on $\mathcal{C}$ if it is lower semi - continuous at each lower limit point of $\mathcal{C}$.

Our first theorem is elementary .

(5.1.7) Theorem: Let $\mathcal{C}$ be a complete chain of subspaces of a normed linear space X and let $T \in B(X)$. Then the map $M \mapsto \| T|M \|$ is a lower semi-continuous map on $\mathcal{C}$.

Proof: Let M be a lower limit point of $\mathcal{C}$. To obtain the result, we need only show that if $\kappa \geq 0$ is such that

$$\| T|N \| \leq \kappa$$

for all $N \in [\ \mathcal{C}\ ,\ M\ ]_-$ then $\| T|M \| \leq \kappa$. So, suppose κ is such that

$$\| T|N \| \leq \kappa$$

for all $N \in [\ \mathcal{C}\ ,\ M\ ]_-$. If $x \in M$ there exists a sequence $\{ x_n \}$ with $x_n \in \cup [\ \mathcal{C}\ ,\ M\ ]_-$ and $\lim x_n = x$. Thus

$$\| Tx \| = \lim \| Tx_n \| \leq \lim \kappa \| x_n \| = \kappa \| x \|$$

and so the result follows .

The next result is a restatement of Lemma (4.2.1) . It is the main motivation behind this chapter .

(5.1.8) Theorem: Let S be a compact operator on an infinite-dimensional normed linear space X and let $\mathcal{C}$ be a subset of Lat S which is a complete chain. If (0) is an upper limit point of $\mathcal{C}$ then the map $M \mapsto \| S|M \|$ is upper semi-continuous at (0) .

Proof: See the proof of Lemma (4.2.1) .

2. Reflexive spaces.

To obtain results about upper semi - continuity of the norm of a compact operator away from the zero space we consider reflexive Banach spaces. We give examples in Section 3 to show that reflexivity is essential in general .

The following two theorems are well - known : see for example [7] or [21] . We need them for what follows .

(5.2.1) Theorem: Let X be a reflexive Banach space. The unit ball of X is compact in the weak topology .

(5.2.2) Theorem: Let X be a reflexive Banach space and let S be a compact operator on X . Then S is a continuous map from the unit ball of X with the weak topology to X with the norm topology .

Proof: See [7 p. 486] .

(5.2.3) Theorem: Let $\mathcal{C}$ be a complete chain of subspaces of a reflexive Banach space X and let S be a compact operator on X . Then the map $M \mapsto \| S|M \|$ from $\mathcal{C}$ to $\mathbb{R}$ is upper semi-continuous .

Proof: Let M be an upper limit point of $\mathcal{C}$. As in Theorem (5.1.7) we need to prove that if $\kappa \geqslant 0$ is such that $\| S|N \| > \kappa$ for all $N \in [\mathcal{C} , M]_+$ then $\| S|M \| \geqslant \kappa$. So, suppose κ is such that $\| S|N \| \geqslant \kappa$ for all $N \in [\mathcal{C} , M]_+$ and consider the set

$$K = \{ x : \| x \| \leqslant 1 , \| Sx \| \geqslant \kappa \} .$$

It follows from Theorems (5.2.1,2) that this set

is itself weakly compact. If $K \cap N \neq \emptyset$ for all $N \in [\mathcal{C}, M]_+$ then, since $\mathcal{C}$ is a chain,

$$\cap \{ K \cap N : N \in [\mathcal{C}, M]_+ \} \neq \emptyset ;$$

that is, $M \cap K \neq \emptyset$. Thus $\| S|M \| \geq \kappa$, and we have the required result .

(5.2.4) Corollary: Let S be a compact operator on a reflexive Banach space X and let $\mathcal{C}$ be a complete chain of subspaces of X. Then the set

$$\{ \| S|M \| : M \in \mathcal{C} \}$$

is closed .

Proof: Let

$$\kappa \in \mathrm{cl} \{ \| S|N \| : N \in \mathcal{C} \} .$$

Then $\kappa = \lim_{n \to \infty} \| S|M_n \|$ where $M_n \in \mathcal{C}$, $(n = 1 , 2 , \ldots .)$.

Let

$$\mu = \{ n \in \underline{N} : \| S|M_n \| \leq \kappa \} .$$

If μ is infinite,let $M = \mathrm{cl} \bigcup_{n \in \mu} M_n$. If μ is finite, let $M = \bigcap_{n \in \underline{N} - \mu} M_n$. In both cases it follows from either Theorem (5.1.7) or Theorem (5.2.3) that $\| S|M \| = \kappa$. But $\mathcal{C}$ is complete, so $M \in \mathcal{C}$.

(5.2.5) Corollary: Let S be a compact operator on a reflexive complex Banach space X and let κ be a real number with $0 \leq \kappa \leq \| S \|$. Then there exist subspaces F and G of X, invariant for S, with $F \subseteq G$, $\dim (G/F) \leq 1$ and

$$\| S|F \| \leq \kappa \leq \| S|G \| .$$

If, moreover, X is infinite-dimensional and $\kappa > 0$ then we may, in addition, require that $F \neq (0)$.

Proof: Let $\mathfrak{M}$ be a maximal chain of subspaces in Lat S. It is easy to check that $\mathfrak{M}$ is complete. If there is a subspace K with $\| S|K \| = \kappa$ then we may let $F = G = K$ and we have the required result.

If not, let

$$F = \mathrm{cl} \cup \{ M \in \mathfrak{M} : \| S|M \| < \kappa \}$$

and

$$G = \cap \{ M \in \mathfrak{M} : \| S|M \| > \kappa \} .$$

Clearly $F \subseteq G$ and by upper and lower semi-continuity we have that

$$\| S|F \| < \kappa < \| S|G \| .$$

Let us suppose that $\dim (G/F) > 1$. Since

$$S_F | G/F$$

is a compact operator, there would exist a subspace H of X, invariant for S, with $F \subset H \subset G$. This would be impossible as $\mathfrak{M} \cup \{ H \}$ would strictly contain $\mathfrak{M}$, thus contradicting the maximality of $\mathfrak{M}$.

For the case when X is infinite - dimensional and $\kappa > 0$ we make use of the main Theorem in requiring that $\mathfrak{M}$ contains an $M \in \mathrm{Lat}\, S$ with $\| S|M \| \leq \kappa$ and $M \neq (0)$.

An illuminating way of looking at this result is the following :

Let S be a compact operator on a complex reflexive Banach space and let $\mathfrak{M}$ be a maximal chain in Lat S . If $0 < \kappa < \| S \|$ but there is no subspace $M \in \mathfrak{M}$ with $\| S|M \| = \kappa$ then there is a discrete "jump" in the chain $\mathfrak{M}$ i.e. there are subspaces F , $G \in \mathfrak{M}$ with $F \subset G$, $\dim (G/F) = 1$, and

$$\| S|F \| < \kappa < \| S|G \| \ .$$

A loose way of putting this is to say that "gaps" in $\{ \| S|M \| : M \in \mathfrak{M} \}$ give rise [redacted] to "jumps" in $\mathfrak{M}$.

This is of special interest since it gives a sufficient [redacted] condition for "jumps" in maximal chains (cf. [19]) .

3. Questions and examples.

Many natural questions arise out of the previous two sections. To give examples, we shall often consider ℓ_1 ,the space of complex sequences which are absolutely summable. That is

$$\ell_1 = \{ \{ x_n \}_{n=1}^{\infty} : \sum_{n=1}^{\infty} |x_n| < \infty \} \ .$$

It is well - known that ℓ_1 is not reflexive .

In this space we let for $k = 1, 2, \ldots$ e_k be the vector in ℓ_1 whose coordinates are all zero except for the kth position where it is one. In other words $e_k = \{ \delta_{kn} \}_{n=1}^{\infty}$.

(5.3.1) <u>Question:</u> In Theorem (5.1.8) can the condition that $\mathcal{C}$ is a subset of Lat S be removed ?

The answer is, in general, no. For an example consider the operator T defined on ℓ_1 by

$$Te_n = e_1 \qquad (n = 1, 2, \ldots\ldots)$$

and let

$$\mathcal{C} = \{ M_n \}_{n=1}^{\infty} \cup \{ (0) \} .$$

where

$$M_n = \overline{\mathrm{sp}} \{ e_n , e_{n+1} , e_{n+2} , \ldots\ldots \}$$

for $n = 1, 2, \ldots\ldots$.

Obviously, $\mathcal{C}$ is a complete chain and (0) is an upper limit point of $\mathcal{C}$. Also T is compact since it is of finite rank . But, $\| T|M_n \| = 1$ for $n = 1, 2, \ldots\ldots$.

(5.3.2) <u>Question:</u> In Theorem (5.1.8) can the requirement that S be compact be weakened to the requirement that S be square - compact and quasi - nilpotent ?

A negative answer may be found in the counter - example to Question (3.3.3) . For this example T is square - compact and quasi - nilpotent. Also, (0) is an upper limit point of the complete chain Lat T but $\| T|M \| = 1$ for all non-zero elements M of Lat T .

We now discuss the relation between the upper semi-continuity of a compact operator and the upper semi-continuity of its square. The following operator and chain of subspaces provides useful examples :

(5.3.3) <u>Example</u>: Let $\{ \alpha_n \}_{n=3}^{\infty}$ be a non-increasing sequence of non-negative reals and let β , γ be reals satisfying

$$0 \leq \beta \leq \gamma \leq \inf \{ \alpha_n : n = 3 , 4 , \ldots \ldots \} .$$

Let T be the operator on ℓ_1 defined by the relations

$$Te_1 = \beta e_1$$
$$Te_2 = \gamma e_1$$
$$Te_n = \alpha_n e_2 \qquad (n = 3 , 4 , \ldots) .$$

Let

$$M = sp \{ e_1 , e_2 \}$$

and, for $n = 3 , 4 , \ldots \ldots$, let

$$N_n = \overline{sp} \{ e_1 , e_2 , e_n , e_{n+1} , e_{n+2} , \ldots \ldots \} .$$

Let

$$\mathcal{C} = \{ N_n \}_{n=3}^{\infty} \cup \{ (0) , M \} .$$

T , being of finite rank, is compact and clearly $\mathcal{C}$ is a complete chain in Lat T with M an upper limit point of $\mathcal{C}$, We compute the norms of the restrictions :

$$\| T|M \| = \gamma \ ,$$

$$\| T|N_n \| = \alpha_n \ , \ n = 3 , 4 , \ldots \ ,$$

and

$$\| T^2|M \| = \beta\gamma \ ,$$

$$\| T^2|N_n \| = \alpha_n\gamma \ , \ n = 3 , 4 , \ldots \ .$$

(5.3.4) Question: Let T be a compact operator on a Banach space X and let $\mathcal{C}$ be a complete chain in Lat T with M an upper limit point of $\mathcal{C}$. If the map $N \mapsto \| T|N \|$ from $\mathcal{C}$ to $\underline{R}$ is upper semi-continuous at M , is the map $N \mapsto \| T^2|N \|$ also upper semi-continuous at M ?

A negative answer comes from taking $\beta = 0$, $\gamma = 1$, $\alpha_n = 1$, ($n = 3 , 4 , \ldots$) in Example (5.3.3) . Clearly, $\| T|N \|$ is upper semi-continuous at M but $\| T^2|N \|$ is not ,

Incidentally [redacted], this example shows that the condition of reflexivity in Theorem (5.2.3) is essential even if we add the additional restriction that $\mathcal{C} \subseteq$ Lat S .

To see this, note that the operator T in the example to Question (5.3.4) is compact and thus that its square is compact. We have already seen that T^2 is not upper semi-continuous .

(5.3.5) Question: Let T be a compact operator on a Banach space X , [redacted] Let $\mathcal{C}$ be a complete chain in Lat T and M [redacted] an upper limit point of $\mathcal{C}$. If the map $N \mapsto \| T^2|N \|$ is upper semi - continuous at M , is the map $N \mapsto \| T|N \|$ also upper semi - continuous at M ?

Again the answer is no, in general. In Example (5.3.3) let $\beta = 0$, $\gamma = 0$, $\alpha_n = 1$ $(n = 3 , 4 , \ldots\ldots)$. $\| T^2|N \|$ is upper semi - continuous but $\| T|N \|$ is not .

While we have Example (5.3.3) at our disposal we ask if Corollary (5.2.4) holds for non - reflexive spaces. Specifically :

(5.3.6) <u>Question</u>: Let S be a compact operator on a Banach space X and let $\mathcal{C}$ be a complete chain of subspaces in Lat S . Is the set $\{ \| S|M \| : M \in \mathcal{C} \}$ always closed ?

Once again, we use Example (5.3.3) to give a negative answer. This time we take $\beta = 0$, $\gamma = 0$, and

$$\alpha_n = \frac{n - 2}{2n - 5} \quad , (n = 3 , 4 , \ldots\ldots)$$

then

$$\{ \| S|M \| : M \in \mathcal{C} \} = \{ 0 , 1 , 2/3 , 3/5 , 4/7 , \ldots \}$$

which is not closed .

Note that this counter - example is stronger than is strictly necessary in that we have required that the chain be contained in Lat S .

4. The square of a compact operator.

Having obtained negative answers to various questions about relating the upper semi-continuity of the norm of a compact operator to the upper semi-continuity of its square, we now come to a positive result on this topic. Firstly we need a lemma :

(5.4.1) Lemma: Let S be a compact operator on a normed linear space X and let $M,, N \in \mathrm{Lat}\, S$ with $M \subseteq N$. Then

$$\| S^2|N \| \leq \| S|M \| \, \| S|N \| + (\| S|M \| + \| S|N \|) \| S_M|N/M \| .$$

Proof: Let $x \in N$, $y \in M$. Then

$$\| S^2x \| \leq \| S^2x - Sy \| + \| Sy \|$$

$$\leq \| S|N \| \, \| Sx - y \| + \| S|M \| \, \| y \|$$

$$\leq \| S|N \| \, \| Sx - y \| + \| S|M \| (\| Sx - y \| + \| Sx \|)$$

$$\leq (\| S|M \| + \| S|N \|) \| Sx - y \| + \| S|M \| \, \| S|N \| \, \| x \| .$$

Taking the infimum over $y \in M$ we obtain

$$\| S^2x \| \leq (\| S|M \| + \| S|N \|) \, d(Sx,M) + \| S|M \| \, \| S|N \| \, \| x \|$$

$$\leq (\| S|M \| + \| S|N \|) \| S_M|N/M \| \, d(x,M) + \| S|M \| \, \| S|N \| \, \| x \|$$

$$\leq [(\| S|M \| + \| S|N \|) \| S_M|N/M \| + \| S|M \| \, \| S|N \|] \, \| x \| .$$

Hence we have the required result .

(5.4.2) Theorem: Let S be a compact operator on a normed linear space X. Let $\mathcal{C}$ be a complete chain of subspaces contained in Lat S and suppose that M is an upper limit point of $\mathcal{C}$. Then

$$\inf \{ \| S^2|N \| : N \in [\mathcal{C}, M]_+ \}$$

$$\leq \| S|M \| \inf \{ \| S|N \| : N \in [\mathcal{C}, M]_+ \} .$$

Proof: The set

$$\tilde{\mathcal{C}} = \{ N/M : N \in \mathcal{C}, N \supseteq M \}$$

is a complete chain of subspaces of X/M and $\tilde{\mathcal{C}} \subseteq \text{Lat } S_M$. $\{M\}$ is the zero- space of X/M and from the hypotheses we deduce that $\{M\}$ is an upper limit point of $\tilde{\mathcal{C}}$. Thus by Theorem (5.1.8) we have

$$\inf \{ \| S_M|N/M \| : N \in [\mathcal{C}, M]_+ \} = 0 .$$

For each $N \in [\mathcal{C}, M]_+$ we have by Lemma (5.4.1) that

$$\| S^2|N \| \leq \| S|M \| \| S|N \| + (\| S|M \| + \| S|N \|) \| S_M|N/M \| .$$

Hence taking the infimum over all $N \in [\mathcal{C}, M]_+$ we have

$$\inf \{ \| S^2|N \| : N \in [\mathcal{C}, M]_+ \}$$

$$\leq \| S|M \| \inf \{ \| S|N \| : N \in [\mathcal{C}, M]_+ \}$$

(5.4.3) Corollary: Let S be a compact operator on a normed linear space X and let $\mathcal{C}$ be a complete chain of subspaces in Lat S. Suppose that M is an upper limit point of $\mathcal{C}$ at which the map $N \mapsto \| S|N \|$ is upper semi - continuous and that

$$\| S^2|M \| = \| S|M \|^2.$$

Then, the map $N \mapsto \| S^2|N \|$ is upper semi-continuous at M.

<u>Proof</u>: This is a simple application of Theorem (5.4.2).

(5.4.4) <u>Example</u>: Suppose that the operator T of Example (5.3.3) satisfies the hypotheses of Corollary (5.4.3). Then $\inf \{ \alpha_n \} = \gamma$ and $\beta\gamma = \gamma^2$. Hence, since $0 \leqslant \beta \leqslant \gamma$, it follows that $\beta = \gamma = \inf \{ \alpha_n \}$ and clearly the map $N \mapsto \| T^2|N \|$ is upper semi-continuous at M as we would deduce from the Corollary.

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